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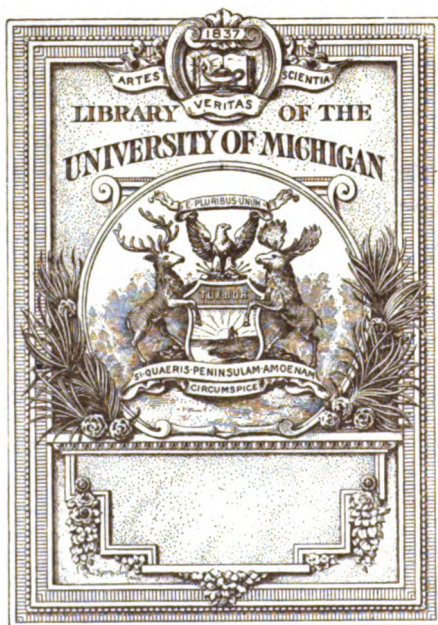
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APPLICATION OF STATISTICAL
METHODS TO THE PROBLEMS
OF PSYCHOPHYSICS

BY

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EXPERIMENTAL STUDIES IN PSYCHOLOGY AND PEDAGOGY

EDITOR:
LIGHTNER WITMER
UNIVERSITY OF PENNSYLVANIA

III. THE APPLICATION OF STATISTICAL METHODS TO THE PROBLEMS OF PSYCHOPHYSICS

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PREFACE

The following study deals with the methods which serve for the determination of the threshold of difference. These methods can be easily adapted to the determination of the maximum and minimum stimulation, so that it seemed justified to refer in the title of the book to the problems of psychophysics in general. These problems are treated by such propositions of the calculus of probabilities as apply to the results of observations of statistical numbers of relative frequency. The term "statistical method" has two slightly different meanings. The first refers to the method of collecting observations on a group of individuals or on one individual at different times. The data obtained in this way must be subjected to some kind of treatment and the term "statistical method" refers in its second meaning to every algorithm for the numerical evaluation of these data or to every definite method by which conclusions can be drawn from them. One may say in general that a problem is treated statistically, if the data are collected with a view to determining certain numbers of frequency which serve as a basis for further deductions.

Some of the ideas expounded in this book date back a considerable time. The considerations of the last chapter originated in the course of a study of some physical phenomena and they were followed up later on in connection with a study of the law of Gompertz-Makeham. Most of the formulae of the third chapter have been known to me for several years, and the idea of a purely formal treatment of the results of psychological experiments came to me in the course of the experiments of Mr. Kobilecki, which were reported lately in the "Psychologische Studien". These small ideas, however, would have remained undeveloped for a long time, had not Prof. Witmer brought to my knowledge an experimental arrangement which could be adapted to the purposes of this investigation. It is so extremely easy to have brilliant ideas in psychology that theoretical discussions justly meet with a certain

distrust, if they are not supported by adequate experimental material. The impossibility of deciding on an experimental procedure which would yield suitable material for the test of the theoretical deductions was the chief reason for not publishing this analysis of the psychophysical methods before. This delay of more than five years proved to be a very fortunate circumstance, not only because the growth of such ideas is very slow, but also because the problems of psychophysics have been the subject of several important publications in the last years. Wundt, Müller, Lipps and Titchener, have treated of the psychophysical methods in the last few years; especially by the work of Titchener is the literature on this problem opened up as it was never before. A similar great help for the analysis of the psychophysical methods was found in Czuber's work on the calculus of probabilities and in those parts of the "Encyklopädie der Mathematischen Wissenschaften" which deal with the calculus of probabilities and its application to statistics. Comparatively little reference to literature is made in this book. Constant reference to work previously done was out of the question, because it would have made the book too voluminous, and occasional quotations are of little or no use. Add to this that the present investigation has not the same starting point as the treatises of the previous authors, and that, therefore, it would have been necessary to argue against views which in themselves are interesting and of merit. This would have given an entirely erroneous impression of my opinion of the value of these views, and hence it seemed best to describe in separate papers the different phases of the development of the psychophysical methods. In so far as the English literature is concerned this presentation will be given in one of the reports on "Die Psychologie in Amerika", which appear from time to time in the *Archiv für die gesammte Psychologie*.

In working out the numerical results it was made a rule to repeat the computation independently whenever it was not possible to apply a thoroughgoing check. The course of the calculations is given in great detail. This was done in order to illustrate the theoretical deductions by numerical examples, and to show that the actual application of the methods described here is simple and that it is shorter than the methods used at present. This

remark refers especially to the method of just perceptible differences, which requires four times as many experiments as the new method to give the result with the same degree of accuracy. Great care was taken to present all the deductions in as simple a form as possible. Mr. Titchener's work has set a standard for such treatment, and it may be hoped that the following considerations will be intelligible to everyone who has gone through the "Manual". This does not mean that all the theorems used are spoken of in this work, but that they are such that they might be understood by every one who can read Mr. Titchener's book. This rule of avoiding complicated deductions made necessary the curtailing of a demonstration in the third chapter at the place where reference is made to Bruns's theorem of the conservation of the $\phi(\gamma)$ -type. The complete solution of the problem would have required a long demonstration of a very technical character, in which some of the more complicated functions are used. It therefore seemed best to give it at some other place.

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PROBLEMS OF PSYCHOPHYSICS

CHAPTER I.

DESCRIPTION OF THE EXPERIMENTS.

The results of a series of experiments on lifted weights are the basis of our discussion. The weights were hollow brass cylinders 7.5 cm. in diameter and 3 cm. in height, closed at one end. The weight of the metal of the cylinders as they came from the shop was slightly less than the smallest weight to be used; by pouring different quantities of melted paraffine into the cylinders sufficient to make each cylinder a little heavier than desired and then scraping out small quantities of the hardened paraffine, the weight could be adjusted very exactly. The exactitude of this adjustment finds a limit only in the exactitude of the balance used. The variations which occur in weights prepared in this way are inconsiderable; they were observed regularly and Table 1 (Appendix p. 173) gives the variations which occurred in winter and spring 1907. The first column of this table gives the number by which each weight could be identified. Since the size and the external appearance of all the weights was the same, it was necessary to distinguish them in some way; this was accomplished by stamping small numbers in the centre of the upper side of every cylinder. These numbers could be seen only on close inspection. The second column of Table 1 gives the weight in grams to which the cylinders were adjusted before the experiments were started; this adjustment as well as the other observations was made with a balance which showed differences of 1 mgr. The differences between the observed weight and the adjusted weight are given in milligrams for every day of control, the dates of which are given at the heads of the columns. The plus sign indicates an

increase of weight, the minus sign a loss of weight. Weights in which a positive or negative difference of more than 10 mgr. from the standard was observed, were readjusted; this, however, happened only three times in the period of three months for which the results are reported. The last column of the table gives the sum of all the variations, understanding by variations the positive or negative difference which the weight has undergone between two observations. The cylinder No. 3, for instance, was adjusted for the weight of 100 gr. and the observed differences from this standard are given in Table 1 as -3, 3, -5 mgr. for the first three observations. This means that on these days the weight of this cylinder was found to be 99.997, 100.003 and 99.995 gr. We have, therefore, an increase of 6 mgr. between the first and the second observation, and a decrease of 8 mgr. between the second and the third. The sum of all the variations does not exceed 31 mgr. for any one of the weights and on the average it is less than 17 mgr. A glance at the table shows that the deviations from the adjusted weight are negative in the majority of cases, but it is worth while noticing that the variations between the single observations were negative in 46 cases and positive in 45 cases, whereas in 23 cases no variations could be detected. The original adjustment of the weights was made in March, 1906. It seems that the cylinders suffer a loss of weight in the beginning, but that they undergo little variation afterwards. The smallness of these variations is chiefly due to the use of anhygroscopic materials, as may be seen by a comparison with the variations of weights which are made of materials more susceptible to hygroscopic influences. For the sake of this comparison Table 2 (Appendix p. 173) is given which contains similar observations for four weights of two patterns. The weights which are called in this table I and II are Cattell weights and belong to the set of weights which were used by Fullerton and Cattell in the series of experiments which are discussed in the publication of these authors.* Weights III and IV are solid wooden blocks filled with lead for adjustment. The table shows the variations of these four

*FULLERTON and CATTELL, *The Perception of Small Differences*, 1892, pp. 116-129.

weights in the period from April 26 till May 10; the upper part of the table gives the results of the observations in grams and the lower part of the table gives the variations between two observations. It may be seen from the table that the variations of the wooden blocks are by far the largest; they are two or three times as large as those of the Cattell weights. These weights are not fit for experimentation which aims at any accuracy not only because the variations are very large, but also because it is difficult to correct a negative variation. The variations of the two Cattell weights are considerably smaller, but the variations of one weight in three weeks are almost twice as great as the variations of all our 18 weights in the course of a little less than three months. It need not be remarked that all the weights were kept under the same conditions, the disadvantage being on the side of our weights which were exposed to rough handling in the course of the experimentation.

These cylinders were not only exactly alike to sight but also, what is perhaps more important, to touch.* The metallic surface of the weights was polished so that it gave the same impression of smoothness no matter where it was touched. Since all the weights, furthermore, were of the same metal and care was taken to keep them under the same conditions of temperature (by keeping all the weights in the same room and by not touching with the warm hand one weight more frequently than the others) an influence of differences of temperature was avoided. The only quality by which one cylinder differed from another was its weight.

These weights were arranged along the circumference of a round turning table at regular intervals, the position of each weight

*The sensations of touch are an important component of the sensation of weight, and for this reason it is necessary to keep them constant, if the purpose of the experiments is an analysis of the sensations of weight. The importance of the sensation of touch for the judgment of the weight of a body was recognized by FECHNER, *Elemente der Psychophysik*, Vol. I, p. 199, and this observation was confirmed lately by A. LEHMANN, *Beiträge zur Psychodynamik der Gewichtsempfindungen*, *Arch. f. d. ges. Psychologie*, Vol. 6, 1906, pp. 446-448, who found that variations of the sensations of touch may cause differences in the estimation of weights, which may be equal to those caused by differences of $\frac{1}{4}$ of the intensity of the stimulus.

being determined by one of the numbers 1-14; a great number of variations in the order of the stimuli may be obtained by interchanging the position of these 14 weights. A general feature of all the arrangements used in our experiments was that weights of 100 gr. were placed at the odd numbers. The weights placed at the even numbers had to be compared with these standard weights. The weights were always lifted with the right hand. The right forearm of the subject rested on a firm table in such a position that the hand from the wrist extended over the edge. The subject was sitting at this table which was placed so that the hand of the subject was vertically over one of the weights of the turning table. The turning table with the weights on it was shut off from the view of the subject by means of a screen. The height of the turning table could be regulated so that the weights were within easy reach of the subject. The lifting was done entirely from the wrist and no part of the arm above this joint was moved. The height of lifting was not regulated, but the subject was instructed to lift the weights in such a way as seemed best in order to obtain an accurate judgment. This was not an equivocal instruction, since every subject was given a large amount of practice before the actual experiments were begun, so that the subject could find the way of lifting which suited him best. The speed of the movements of the hand did not seem to vary much after the subject had settled down to a certain way of lifting. Most subjects favored an excursion of the hand of approximately 5-7 cm.; one subject preferred to lift his hand considerably higher, and another adopted a very small excursion of 2-3 cm. This is in so far of importance as the time of the movements was kept constant and the excursion of the hand gives an indication of the velocity with which the weights were lifted. The height of lifting depends to some extent on the flexibility of the hand at the wrist.

The movements of the hand were regulated by a metronome of Mälzel, which beat 92 times a minute and every fourth beat of which was marked by the stroke of the bell. On the stroke of the bell the hand of the subject went down and grasped the weight; on the second beat the hand was lifted; on the third

beat the weight was put back on the table, and on the fourth beat the hand returned to its original position. After the weight was put back the operator had sufficient time to turn the table and to bring the next weight directly below the hand of the subject. The experiments thus could go on without interruption and were continued for five complete turns of the table, so that in such a series five judgments were given on every pair of comparison weights. The position of the hand of the subject was the same in lifting all the weights.

The judgments referred to the second weight of the pair. The possible judgments are that the second weight is heavier or lighter than the standard, or that both weights are equal. The subject was allowed to express the degree of his confidence by annexing one of the numerals 1, 2, or 3 to his judgment. The unit indicated an ordinary degree of confidence and was omitted, 2 and 3 indicated higher and highest degrees of confidence, but it was found that the subjects rarely went beyond 2. For the so-called doubtful and equality cases the following arrangement was made. It is a fact that those cases where the judgment is absolutely doubtful diminish in number when the practice of the subject is increased.* These cases alone ought to be called doubtful cases, because in the other class of cases the judgment is "equal" and the judgment is not doubtful at all. Cases where the subject was unable to form a judgment occurred in the preliminary series, but they were rare in the actual investigation and the experiments were repeated whenever this happened. When no difference could be detected the subject was instructed to make a guess whether the second weight was heavier or lighter, but to mark these cases by the words "heavier guess" or "lighter guess." The "guess"-judgments are theoretically, i. e. by their definition, identical with the equality cases, but practically we met with some difficulties. Most subjects had in the beginning of their training some difficulty in distinguishing the "guess"-judgments from judgments which expressed

*Cfr. WUNDT, *Physiol. Psychologie*, 5 ed., 1903, Vol. 1, p. 482; G. E. MUELLER and FR. SCHUMANN, *Über die psychologischen Grundlagen der Vergleichung gehobener Gewichte*, *Arch. f. d. ges. Physiologie*. Vol. 45, 1889, p. 40

the recognition of an existing difference with a very low degree of certainty. Later on all the subjects but one (subject III) found no difficulty in separating these two classes of judgments. This subject almost never seemed to have the impression of equality of the stimuli and he consistently gave "guess"-judgments when he did not feel quite sure; in these cases he mostly gave the judgment "heavier guess". Since it was not possible to eradicate this habit by a very considerable amount of practice given to this subject, it was decided not to interfere with his way of judging.

All the judgments given were recorded. For this purpose special record sheets were prepared which contained seven rows of five places each. Every comparison weight was given one line, so that the 35 experiments of running the table around five times could be recorded on one blank. If the comparison weights are written down in the same order in which they are placed on the table, every column of the record gives the experiments of one turn of the table and the rows contain the judgments given on one comparison weight in the five turns of the table. The following example will make clear this way of

104	h	h	h_2	h	h	5
92	h	hg	l	l	hg	3
108	h_2	h	h	h	h	5
88	l	hg	l	l	l	1
96	h	h	h	hg	h	5
100	hg	h	h	h_2	h	5
84	hg	l	l	l	l	1
	6	6	4	4	5	

keeping the records. The letters h and l stand for the judgments "heavier" and "lighter." The "guess"-judgments were

recorded by hg and lg, according to whether the judgment "heavier guess" or "lighter guess" was given. The first column of the record gives the comparison weight; it is not necessary to put down the standard weight because it is the same for all the pairs. In the 35 spaces of such a table of five columns all the judgments given during the five turns of the table may be recorded by entering every judgment as it is given in its place running from the top of the columns downward. The numbers in the last column refer to the number of "heavier"-judgments given on this weight, no discrimination being made between "hg" and simple "h"-judgments. The numbers at the bottom of the columns do the same for the columns i. e. they give the number of "heavier"-judgments given in one turn of the table without distinguishing between guesses and ordinary judgments.

This experimental arrangement made it necessary to have three persons present: The subject, the operator of the table and the recorder. The recorder of course became acquainted with the arrangement of the comparison stimuli, and it was found very soon that this knowledge was of disturbing influence, if the recorder had to serve as subject in the series where the same arrangement was used. For this reason nobody was used as recorder in a series before he had finished it as a subject; as a rule the recording was done by the conductor of the experiments, who could not avoid having some knowledge of the order of the comparison weights anyhow, or by an assistant who did not act as a subject. In this way it was tried to keep from the subject all knowledge about the actual relation of the stimuli, and the experimental arrangement fairly excludes the possibility of the subject's unwittingly learning what this relation is. Even the knowledge of the order in which the pairs are presented does not interfere with the proper performance of the experiments, if attention is devoted to the experiments and not to the order of the succession of the judgments. There were some cases—very few in number—where the subject reported that the idea was impressed upon him he knew the order of the pairs. Cases of this type were investigated on the spot and it was found that this impression of the subject was only seldom justified. These

cases, as a matter of fact, are perhaps more interesting as examples of memory illusions and of the origin of faulty impressions based on incomplete observations. After the series reproduced above, for example, the subject reported having been conscious that there was in the series a succession of weights described by the judgments h_2 , l , hg , and the subject offered to pick out these weights. A number of trials was immediately given, but the subject could not make up his mind and finally gave up this task as impossible. Then the record of this series was surveyed, and it was found that a succession of judgments as described by the subject did not occur a single time and that the h_2 -judgments were given on different weights. This example points out two facts: 1, that memory is very unreliable for events on which attention is not fixed and 2, that the degree of confidence with which a judgment is given does not depend solely on the amount of objective difference, because the degree of confidence with which the judgments on a certain pair of comparison weights are given varies considerably. The latter fact is well illustrated by a few cases where h_2 and l_2 -judgments were given on the same pair of comparison weights inside the same series. These cases were

Subject St., April 2, 1906, I A, No. 8 (h_2 and l_2 on 96);

Subject W., June 8, 1906, IV A, No. 4 (h_2 and l_2 on 100);

Subject B., Oct. 23, 1906, I, No. 2 (h_2 and l_2 on 108).

Great confidence or lack of confidence are no criteria of the reliability of the judgment, and it is not likely that the confidence in the judgment is in a simple relation to the difference between the stimuli. There are cases on record where the subject complained of very little or no confidence at all, and where a survey of the results shows that the judgments in this series have fewer mistakes than any other series of the same day. In some cases, on the contrary, this lack of confidence

was in so far justified as the results showed a considerable number of erroneous judgments.*

There are two points in our experimental procedure which need some explanation; the first is the regulation of the movement of the hand by the metronome, the second is the taking of the experiments in series of 35. The regular sound stimuli may have some influence on the judgment, and the length of one series may tend to produce states of fatigue or variations of attention which make results of different parts of one series not directly comparable. It seems that only the second possibility requires special attention, because the use of the metronome for regulating the movements of the hand has been found advantageous and unobjectionable in several previous investigations. It will be shown in the second chapter that there exists no evidence of an appreciable influence of making the experiments in series of 35. There are several ways in which the beats of the metronome might possibly influence the judgments:

(1.) Regular acoustical stimuli produce a feeling process† which may be the immediate cause for judgments in a certain direction. It is very easy to observe this waving up and down of feelings when listening to the beats of a metronome, but these feelings are by far less noticeable if attention is primarily concentrated on the performance of the experiments.

(2.) The beats of the metronome may influence attention, which is the psychical function affected by the rhythmic motion of lifting the weights.

*The first part of this observation contradicts the statement of Fullerton and Cattell that with an increase of the difference of the intensities of the stimuli our judgment changes from complete uncertainty to confidence. Judgments on the same amount of objective difference are given with a very variable degree of confidence. Of course it is possible to say that on the average our confidence increases with the difference of the intensities, but taking such an average is not a well defined process. The latter part of the observation agrees with a remark of these investigators (*On the Perception of Small Differences*, 1892, p. 126 sqs.) in so far as it shows that accuracy is not necessarily proportional to subjective certainty, an observation to which these authors give the piquant turn that those subjects who are most confident are the least accurate of all.

†W. WUNDT, *Phys. Psych.* 5 ed., 1903, Vol. 3, p. 23.

(3.) The acoustical stimuli may interfere with the mechanical contraction of the muscles in such a way as to reinforce or inhibit the action of the muscles. The observations of Bowditch and Warren,* Cleghorn,† Hofbauer‡, Yerkes¶ and others show that the effect of two stimuli which are applied in not too great an interval is never equal to the sum of the effects of the stimuli, but that they interfere in such a way as to inhibit or reinforce each other. There is little doubt that there is the possibility of such an influence of the regular acoustical stimuli on the contraction of the muscles, although one can not say at present what effect on the judgment of weight such an interference of stimuli may have.

We have more positive information in regard to the second point, which was the object of an experimental investigation of Smith.§ For our present purpose, however, it is less important to have an analysis of the factors which decide the judgment on the weight of a body, than to arrange the experiments in such a way that the conditions for all the judgments are the same, no experiment being favored by an influence which is not at work in the others too.

Similar considerations hold good for the possibility of an influence of the length of a single series. Besides fatigue and shifting of attention there is the possibility that the stimulation produced by the first lifting does not die out immediately, but that it interferes with the next stimulation, so as to inhibit it or reinforce it. This possibility was considered recently by Lehmann.|| This author supposes that such an influence exists and he gives

*H. P. BOWDITCH and J. W. WARREN, *The Knee-Jerk and its Physiological Modifications*, *Journ. of Physiology*, Vol. 9, p. 60, 1890.

†A. CLEGHORN, *The Reinforcement of Voluntary Muscular Contractions*, *Amer. Journ. of Physiology*, Vol. 1, p. 336, 1898.

‡L. HOFBAUER, *Interferenz zwischen verschiedenen Impulsen im Centralnervensystem*, *Arch. f. d. ges. Physiologie*, 1897, Vol. 68, p. 546.

¶R. M. YERKES, *Bahnung und Hemmung auf tactile Reize durch akustische Reize beim Frosche*, *Arch. f. d. ges. Physiologie*, Vol. 107, 1905.

§MARGARET KEIVER SMITH, *Rythmus und Arbeit*, *Phil. Stud.* Vol. 16, 1900, pp. 71-133, 197-305.

||A. LEHMANN, *Beiträge zur Psychodynamik der Gewichtsempfindungen*, *Arch. f. d. ges. Psychologie* Vol. 6, 1906, pp. 425-499

a mathematical expression for it. An influence of the type Lehmann describes will not impair the experiments but it will go in as a constant factor. Indeed since the interval between two successive stimulations is constant, no matter whether the preceding stimulus inhibits or reinforces the following, this influence will tend to produce a state where the amount of inhibition or reinforcement is the same for all the weights. This influence, therefore, will be distributed equally over all parts of the series of experiments.

The first weight in every pair was 100 gr.; it was compared with weights of 84, 88, 92, 96, 100, 104, 108 gr. in those series the results of which are discussed here. It will be noticed that the series of comparison weights is not equally extended above and below the standard. The motive for the choice of this series of comparison weights was that it was observed in the preliminary experiments that all the subjects had the tendency to give a majority of "heavier"-judgments on the comparison of 100 with 100. It does not seem profitable to use an equal number of positive and negative differences, because one would get a very great percentage of "heavier"-judgments in the upper part of the series and a small percentage of "lighter"-judgments in the lower part, or one would suffer from the opposite inconvenience as the case may be one of overestimation or underestimation of the second stimulus. It came out as a result of our experiments that for some purposes, e. g. for the determination of the threshold by the method just perceptible differences, the use of at least one weight beyond 108 gr. is desirable, and that with some subjects the use of a weight as far down as 84 may be dispensed with.

Presenting these seven pairs of stimuli five times to the subject in the way described takes approximately three minutes (3' 8"). After the series was finished the subject was given a rest of some five minutes before another series of experiments was begun. This arrangement enabled us to make six or seven series of experiments in the course of one hour. It was, therefore, possible to obtain in one hour of experimentation 30 to 35 judgments on each one of the pairs. The order in which the pairs were presented was not changed before 100 judgments on every difference were obtained. Such a group of 100 experi-

ments was marked with a Roman numeral. It furthermore was found convenient to divide the results in groups of 50 experiments, which were marked by the corresponding Roman numerals and by the Arabic numerals 1 and 2; III2, thus, refers to the group of experiments which comprises those from the 51st to the 100th experiment inclusive in the third series.

The data of five different series were used for the present investigation. Two of these series were made in spring 1906, the others in the year 1906-07. The order of the pairs was this:

Ia	96,	104,	108,	84,	92,	100,	88;
IVa	104,	92,	108,	88,	96,	100,	84;
I	96,	104,	108,	84,	92,	100,	88;
III	84,	104,	96,	100,	92,	108,	88;
IV	84,	88,	92,	96,	100,	104,	108.

The series made in spring 1906 are marked by the letter a.* Nineteen subjects were experimented on, but only the results of those are used for the purpose of the present study who have gone through series I, III and IV. The number of these subjects was seven, three of whom also made series Ia, and IVa. The subjects will be spoken of as subjects I, II, III, IV, V, VI, VII. With the exception of subject V all were males. Subject III was 42 years; the age of the others was between 21 and 30. Subjects I, II and III have gone through all the series and we have for each one of them 450 judgments on every pair of comparison weights, since only the second part of series Ia was available.† The other subjects gave 300 judgments on every pair. The total number of experiments for each one of the first three subjects is 3,150, and for each one of the others 2,100, so that our discussions are based on the results of 17,850 experiments. This number is great, but by no means excessive; it will be seen that it is sufficient for most purposes.

*The plan of these series of experiments was the outcome of several preliminary investigations undertaken by the Psychological Laboratory of the University of Pennsylvania. They were conducted by Professor Witmer, from whom the records were obtained for use in this investigation.

†The results of Ia1 show the influence of practice and were, therefore, not used for the purpose of this study. Series IIa, IIIa and IVa contain experiments which are not comparable to those of the other series, because the experimental conditions (time of lifting, interval between the pairs, order of standard and comparison weight) were different.

It is easy to see that the arrangement of our experiments is a modification of the one described by Martin* and more recently used also by Smith,† the modifications having the purpose of eliminating the space error. The principles for the choice of this experimental arrangement were laid down by Müller and Schumann, whom it seemed safe to follow in this respect. The difference of the arrangement used by Martin and by Smith from the one used in our experiments consists chiefly in that that these investigators used a stationary table and only one standard. Having the weights placed on a stationary table implies that the so-called Fechnerian space error is taken in, whereas it seemed desirable to avoid this error in our experiments. The use of only one standard weight causes a rise in the temperature of the metal owing to the fact that this weight is touched very frequently with the warm hand. This is to be avoided for two reasons: First, because differences of temperature might influence in some way our judgments of weight, and second because the standard weight can be recognized by being warmer,‡ which introduces an almost incontrollable influence. The use of only one standard weight may be less objectionable if the original Cattell weights or other weights are used which are made of materials which are bad conductors of heat. In experiments in which the turning table is used, it is almost indispensable to use several standard weights.

The purpose of our experiments was in so far different from that of Martin and Müller, as we wanted to have as few disturbing influences as possible and Martin and Müller were chiefly interested in the analysis of the time and space error. It seems that the time error may be kept constant by

*MARTIN und MUELLER, *Zur Analyse der Unterschiedsempfindlichkeit*, 1899, p. 3 sq.

†MARGARET KEIVER SMITH, *Rythmus und Arbeit*, *Phil. Stud.* Vol. 16, 1900, pp. 96-98.

‡A casual observation of the increase of the temperature of the standard weight by frequent contact with the skin was made by MUELLER and SCHUMANN *Über die psychologischen Grundlagen der Vergleichung gehobener Gewichte*, *Arch. f. d. ges. Physiologie*, Vol. 45, 1893, p. 112) who, however, did not change their experimental procedure.

means of regulating the movement, whereas a similar supposition can not be made for the space error unless very definite instructions are given, which it is hard to follow strictly in an extended series of experiments. For this reason it seemed best to take in the time error as a whole, but to avoid the space error. The question whether the time error may be eliminated by an appropriate arrangement of the computation is of secondary nature, and one may doubt whether there is any great use in doing so. In this respect it seems that our way of experimenting is a slight improvement over Müller's method, in so far as it makes it possible to avoid the space error without further inconvenience and to perform the experiments in comparatively short time. It is not possible to pass an equally favorable judgment on our modification of the series of judgments of which the subject has the choice. It has been mentioned that one subject could not be trained to the proper use of the "guess"-judgments and it seems to come out as a result of the experiments that the guesses are more likely to be correct than incorrect. This indicates that the "g"-judgments were also for the other subjects not equality judgments in the proper sense of the term, meaning that the subject had nothing in his sensation to go upon in the formation of his judgment, but that they were judgments of a low degree of certainty. This is a fact which, perhaps, is not void of interest but which has no direct bearing on the problem of measuring the accuracy of sensations and, after having verified this fact, it will be best not to use this distinction between "heavier guess" and "lighter guess". The only natural judgment in these cases is that of equality between the two stimuli. The possibility of giving a "guess"-judgment aids the subject in his natural tendency not to commit himself.

It seems that a similar remark must be made in regard to letting the subject express the degree of confidence with which the judgments are given. The meaning of the terms "an ordinary degree of certainty," "a high degree of certainty" and "a very high degree of certainty" seems to be clear at first and not requiring any special definition. In the actual experimentation one finds very soon that it is difficult to give these terms a definite meaning, an observation which was also made by

one of Müller's best trained observers.* It seems that there is some danger of mixing up two different problems, the problem of the degree of subjective confidence and that of the accuracy of sensation. It certainly would be not only an interesting but also a very important problem to find their relation, but it seems that the factors which determine the degree of subjective certainty are not yet as well understood as they might be, and that other experiments might be used more profitably for settling this problem. The problem of finding a measurement of the accuracy of sensations is difficult enough and it need not be made more complicated by attacking it from the most difficult side. It seems advisable for this reason not to use other judgments than "heavier," "lighter" and "equal," unless one intends to connect the study of the exactitude of sensations with some other problem. The degree of subjective confidence with which the judgments were given was disregarded entirely in the working out of the results for the present study.

The Tables 3-9 (Appendix pp. 174-177) contain the results of the experiments. Each subject is given a separate table which contains the judgments for all the seven pairs of comparison weights for every series separately. The numbers in the column T are the numbers of "guess"-judgments which are the sum of the "hg"- and the "lg"-judgments. The numbers under the heading "h" and "l" give the number of "heavier" and "lighter"-judgments. The sum of the numbers T, h, and l in one line of a column is equal to 50. The different series are marked in the way explained above. The numbers at the bottom of the columns in the row marked Σ give the sum of the numbers in a column.

The numbers of these tables, as they stand here, illustrate plainly the one fact that the results of psychological experiments show a great variability. The greatest amount of care was taken to keep all the conditions of the experiments constant and there arise, nevertheless, variations of very considerable size. This indicates on how terribly uncertain a basis most of the results rest which are gained from psychological experiments.

*MARTIN und MUELLER, I. c. p. 9.

Publications are not scarce which claim to demonstrate the influence of a certain factor on a psychical phenomenon by observed differences in relative frequencies amounting to 4 or 5%. It is very obvious to raise the objection to these investigations that a similar or perhaps even a greater variation of the results might have been obtained, if the experiments had been repeated without varying the condition the influence of which was to be demonstrated.

The next observation is that there does not exist apparently a difference of the intensities which is always judged in the same way. A negative difference of 16 grams is by far beyond the difference which is called the threshold, but there occur some cases in all the subjects where this difference could not be detected or was judged the wrong way. A difference which is such that greater differences are always judged in the same way is a fiction, the fact is that the judgments of one subject on the same difference vary, and it is impossible to foresee in which judgment a certain experiment will result. This is not a new observation as e. g. Müller and Cattell* in their discussions of the notion of the threshold have laid stress on the fact that a difference which is always judged in one way does not exist. Lately the view was taken by Kobilecki† that "there must be something wrong with the experiments" if this occurs. This view, if once taken, is of course irrefutable, because it lays down as a criterion of the correctness of the experiments, that they must conform with the requirement of a difference of intensities which is always judged in the same way, e. g. that it is always possible to find a difference which is judged equal. The only objection is that it is not possible to find another fault with the experiments in question than that their results do not agree with one's definition of a correct experiment, and that it does

*There is a slight difference in the positions of Cattell and Müller. Müller takes it as a given fact that there does not exist a difference which is always judged in the same way, whereas Cattell assumes the suppositions of the theory or errors and argues that the assumption of such a difference is contradictory.

†STANISLAUS KOBILECKI, *Über die Wahrnehmbarkeit plötzlicher Druckänderungen*, *Psychologische Studien*, Vol. I, 1906, p. 293, where the reference to Müller's writings may be found.

not seem to be quite justified to discard experiments which were performed with just the same care as others, merely because their outcome was not such as it ought to be according to a definition laid down beforehand.

These facts seem to suggest the following view. If the subject is required to give judgments on the comparison of two stimuli his judgments will vary, so that one can not possibly foresee what the judgment will be in any particular experiment. The only thing we know beforehand is that, if the experiment succeeds, the judgment will belong to one of those classes of judgments which were admitted. This is the formal character of random events and we introduce the notion that there exists a definite probability for every class of judgment, that the comparison of certain two stimuli by a given subject under well defined conditions will result in a judgment of this class. If we admit the judgments "heavier", "lighter" and "equal" we will have the probabilities p , q , r that the comparison of a given difference will result in the corresponding judgment. The numerical values of these probabilities may vary from subject to subject, and it is a problem of investigation whether they remain constant for the same subject and the same difference, or whether they are subjected to temporal variations.* For

*Wreschner defines "trustworthiness" of a sensation (Zuverlässigkeit) as the expectations which a single sensation or a judgment on it has to be confirmed in the case of repetition under conditions which are as nearly alike as possible. (*Methodologische Beiträge zu psychophysischen Messungen, Schriften der Gesellschaft für psychologische Forschung*, 1895, p. 25 "Hierunter (unter Zuverlässigkeit) wollen wir die Aussichten verstehen, welche eine einmalige Empfindung oder deren Beurtheilung hat, in einem Wiederholungsfalle, der unter möglichst gleichen Versuchsbedingungen stattfindetbestätigt zu werden...."). The correct analysis of this notion, of which Wreschner makes but little use, might have led to the introduction of the notion of a probability of judgments of certain type. Wreschner's definition refers merely to the subjective side of the notion of a probability, neglecting the objective definition which alone is suitable for the mathematical treatment. For the explanation of a mathematical probability as a measurement of the degree of certainty see v. KRIES, *Die Principien der Wahrscheinlichkeitsrechnung*, 1886, and STUMPF, *Über den Begriff der mathematischen Wahrscheinlichkeit*, *Berichte der bayrischen Akademie, Phil. Kl.* 1892.

positive differences which are considerable the value of the probability of a "greater"-judgment comes very near to the unit and the probabilities of other judgments correspondingly are very small. For negative differences which are considerable the probability of a "smaller"-judgment is only little different from the unit and the probabilities of the judgments "greater" and "equal" are very close to zero. Data like those embodied in the Tables 3-9 are observations on the repeated realization of chance events and they may be used as empirical determinations of the underlying probabilities. The question is, how far do the results comply with this notion of a probability of a judgment of certain type, and how much can be done with it for the solution of the problems of psychophysics.

CHAPTER II.

ON THE STATISTICAL NUMBERS OF RELATIVE FREQUENCY.

A statistical investigation may be confined to a description of the conditions and a statement of the facts which were observed. Such a statement is in itself valuable and not void of interest because it contains the description of a fact, but as long as this fact is not connected with other facts its statement is not so much knowledge as the material for the future acquisition of knowledge. On this ground one even can not conclude that under similar conditions results will be obtained which resemble those of the first series of observations. It is, indeed, out of question to reproduce exactly the same conditions and, since one does not know anything about the conditions which necessitate the result, one can not positively say that only the observed conditions are of importance and one must resign the hope to foretell future results. But the main interest of all investigations is to know, whether the same, or at least similar results will be obtained in a future repetition of the observation. Before such a statement can be made it is necessary to form one's views about the causes which were at work to produce the first result. This can be done by demonstrating the causal relation between the results of the observation and a certain group of conditions by means of the method of experimental variation of the conditions. The application of this method is comparatively simple in those cases where the conditions under which we want to observe a phenomenon are well under our control. This method becomes more laborious if our control of the conditions is not complete enough to enable us to establish causal relations; in these cases it becomes necessary to introduce new notions.

One forms the hypothesis that the result is due to a complex of conditions which remain constant and another group of conditions which are variable. The variations of the latter group

are supposed to have random character, i. e. they do not show a recognizable law or regularity. The group of constant conditions is represented by those conditions which one can vary or keep constant at will, or which by their nature do not vary, or are supposed not to vary. The result of these constant and variable conditions, then, is determined in so far as it must have a certain general character, but it is subjected to quantitative or qualitative variations. The constant conditions determine not only the class, to which an event which depends on them belongs, but they determine also the amount of variation which may occur in individual cases. Degrees of variation may define classes which are sub-classes of the class. Individuals of these sub-classes occur with different relative frequency, which is a fraction, the denominator of which gives the number of all individuals of the class, and the numerator the number of the individuals of the sub-class. This number, which is smaller than one, has the character of a mathematical probability. Every group of conditions which gives certain numerical probabilities to every sub-class is called a system of causes, or abbreviated a cause of an event of the class. Different systems of causes produce different results in so far as they may 1, produce events which belong to entirely different classes, or 2, that they give numerically different probabilities to the sub-classes of the same type of events. This terminology differs from the common usage of the word cause which designates a group of conditions which necessarily produces a certain event; here we mean by this word a group of conditions which gives a certain probability to the event in question. This group of conditions may or may not be susceptible of further analysis, but as long as further information is not supplied, it is characterized only by the numerical value of the probability of the event. Two groups of conditions which give the same probability to the same event are not distinguishable on this basis.* If we know that in a series of observations on a certain event the result of a single observation

*This may be illustrated by the following example. It was found in an investigation on the estimation of long time intervals by a great number of untrained subjects, that zero was a favored numeral occurring at the last place much more frequently than any other numeral. An other investiga-

depends on a system of causes which is characterized by the numerical value of the probability of the event, we are not only able to state the most probable outcome of the series, but we also can assign a definite probability with which we may expect a given deviation from the most probable result. We base our expectation that a repetition of the series of experiments will give similar results on the identity of the conditions which we know and on the supposition of the random character of the influences which we do not know.

Repetitions of the same series of observations, however, do not give exactly the same result and the question arises necessarily, whether the differences between these results are due to chance variations or whether they are due to a change in the underlying system of causes. This question, as a matter of fact, is the all important problem of every statistical investigation which applies the method of experimental variation of the conditions. A certain phenomenon, e. g. a mental state, is observed under certain conditions and from these observations statistical numbers of relative frequency are derived. One or several of the conditions which we have so far under control as to vary them at will may be susceptible of qualitative or quantitative variation. Keeping all the other elements constant we vary these conditions and we want to know, what the influence of this change is. This influence is characterized only by the differences of

tion on the estimation of short time intervals by a small number of highly trained subjects showed a similar great frequency of zero. If nothing else had been known than the percentages expressing the great frequency of zero, one had been obliged to conclude that similar conditions were at work in both cases, the conclusion of identical conditions being made impossible by differences in the numerical values of the percentages. Collateral evidence, however, showed that the conditions involved were absolutely different. (See the author's article on "*Systematic Errors in Time-Estimation.*" *Amer. Journ. Psychology*, 1907, Vol. 18.) The collateral evidence in this case had the form of differences in the percentages of the other numerals, especially of 5 and this circumstance indicated the possibility and necessity of further psychological analysis. It is not necessary that the collateral evidence must have the character of statistical numbers of relative frequency; introspective data or our general knowledge of the processes involved may make one hypothesis more likely than the others.

the observed numbers of relative frequency in the two series. The influence of the variation of the condition is demonstrated, if we can convince ourselves that the difference is due to a different numerical value of the probability of the event in the second series and not to chance variations. It is the object of further consideration to form a view on the nature of this influence, and it will depend on the particular kind of problem one has to deal with which views will seem acceptable. The real difficulty of the treatment of the statistical data consists in the demonstration that the difference of results is due to a different system of causes. The outcome of the comparison of the two series consists in the difference of certain numerical values, and similar differences would have been found if the conditions had not been changed. Are these differences then to be attributed to chance variations, or are they due to differences in the objective conditions?

The answer usually given is that the differences must be accounted for by a change of the objective conditions, if they are unequivocal and considerable. By the term "unequivocal" one means that the differences must always have the same sign i. e. there must be uniform decrease or uniform increase, but not increase alternating with decrease, if the group of conditions which we investigate varies in one way. One finds it rather hard to comply with this condition in actual investigations and one must not be too strict in the application of this requirement, because even with phenomena which are as constant as the size-weight illusion one finds exceptions. As a rule one will be satisfied if the percentage of those cases which do not comply with the requirement of an unequivocal variation is small. A similar vagueness lies in the requirement of considerable differences. It is a matter of course that one can not speak of absolute differences but only of relative or percentual differences. The absolute differences between the results of two statistical series must increase if the number of observations becomes greater, even if we have to deal with no other but chance variations, as it follows from the theorem of Bernoulli. A difference which may be considerable in one case will be inconsiderable in another case and, furthermore, there exists no fixed limit where a nu-

merical difference begins to be considerable. The notion of a considerable difference is too indefinite to be serviceable for distinguishing between two results.

The particular character of the notions used does not permit a strict answer, but only the statement of the view which is more likely, because it has the greater probability. On the basis of the results one may form a number of different hypotheses which may explain both results but which have a different degree of probability. It is obvious at once that no difference between the results, no matter how great it may be, can be a strict demonstration of a difference in the underlying systems of causes. Since the result of a single experiment is not strictly determined as far as our knowledge goes, it follows that one has only a certain probability with which one may expect the result to fall within given limits, and it is always possible that the opposite of an event which is only probable arrives, no matter how near to the unit the probability may be. Deviations of every size are possible, only the probabilities of their occurrence are different.

The first requirement for the prediction of future results is that the observed numbers of relative frequency have not only the formal but also the material character of mathematical probabilities. Such a number has the formal character of a probability, if it is a fraction the denominator of which gives the number of all the cases and the numerator the number of all those cases which are favorable to the event. The ratio of the number of the judgments "heavier" to the total number of judgments given on a pair of weights has the formal character of a mathematical probability. The percentage of these judgments, and the ratio of the number of "heavier"-judgments to the number of "lighter"-judgments have the character of functions of mathematical probabilities ($100p$ and $\frac{p}{1-p}$). Exam-

ples of numbers of relative frequency which have not the formal character of mathematical probabilities, nor of functions of such, are the ratio of the number of births which take place in a country during a year to the number of inhabitants of this country, or the ratio of those men of a country who choose a certain

profession to the number of inhabitants.* Numbers of this kind must not be treated as mathematical probabilities and certain important conclusions cannot be drawn from them.

Neither the theorem of Bernoulli for constant probabilities, nor the theorem of Poisson for variable probabilities can avail for demonstrating that a statistical number of relative frequency has the material character of a probability. The theorems of the calculus of probabilities, indeed, do not contain any statement about actual events, but they give only the formal character of those events to which the attribute of randomness belongs. The only possible way of demonstrating that a statistical number of relative frequency has the material character of a mathematical probability is to show that this number approaches in a certain way a definite limit, if the number of observations is increased indefinitely. Let us suppose we have n series of observations on an event E , each one comprising s observations of which $m_1, m_2, \dots, m_n$ have given the result E . Then the first requirement says that the fractions

$$\frac{m_1}{s}, \frac{m_2}{s}, \dots, \frac{m_n}{s}$$

must be grouped around a number p in a certain way. It follows from the fact that p is the limit of the numbers $\frac{m_i}{s}$ that they will be clustered around p more closely than around any

*For use of the birth rate see LAPLACE, *Théorie analytique de la probabilité* art. 30, 31. Statistical numbers of the second type were used by Mr. CATTELL in his study of American men of science (*A Statistical Study of American Men of Science, III*; *Science*, December 7, 1906, N. S. Vol. XXIV, No. 623.) These numbers have not the formal character of a mathematical probability because among the men living in a country there are also those who have chosen their career, so that the number of inhabitants does not represent a totality of cases which may go one way or the other. The choice of profession of all the men born in a country during a certain period of time represents more nearly a set of cases which depend on chance. It is not likely that such a number would have the material character of the probability of becoming e. g. a scientist for a child born in this country, because the denominator of such a ratio ought to be the number of all those circumstances which decide the choice of a man's life work, a totality of cases to which no definite number can be assigned.

other value. As to the law of the distribution of these numbers there are two cases possible which are fundamentally different for the interpretation of the event E: either the distribution follows the $\Phi(\gamma)$ -law or it does not. The interpretation will be different according to the following cases.

(I.) The distribution follows the $\Phi(\gamma)$ -law.

1. The distribution is such that the number of deviations, the absolute value of which is smaller or equal to γ , γ being any value,

$$\left| \frac{m}{s} - p \right| \leq \gamma \quad (i = 1, 2, \dots, n)$$

is given by

$$n \Phi(\gamma) = \frac{2n}{\sqrt{\pi}} \int_0^{h\gamma-x^2} e^{-x^2} dx, \quad (1)$$

where n is the number of series of observations and where h is a constant called the coefficient of precision which is given by

$$h = \sqrt{\frac{s}{2p(1-p)}}. \quad (2)$$

This is the case of a distribution according to Bernoulli's theorem. This will be e. g. the distribution of observations, where the single observation consists in drawing a ball from an urn which contains black and white balls only, if the ball is put back after its color was registered. If p is not known one must take the arithmetical mean of the numbers $\frac{m}{s}$.

$$a = \frac{1}{n} \sum_{i=1}^n \frac{m_i}{s} = \frac{1}{ns} \sum m_i. \quad (3)$$

and one obtains for the corresponding coefficient of precision

$$h' = \sqrt{\frac{s}{2a(1-a)}}. \quad (4)$$

Statistical events of this type are similar to the drawing of balls from an urn which contains the same number of black balls and white balls throughout the series. The results of such a series may be regarded as the products of a constant system of underlying causes represented by those conditions which determine the probability of the event, and of another system which is subjected to chance variations. A statistical series of this kind is called by W. Lexis* a typical series of normal dispersion. We are allowed to suppose that a repetition of the observation will give a result which complies with the expectations which we base on the calculus of probabilities.

(2.) The distribution follows the $\Phi(\gamma)$ -law but the precision required by formula 1 is not equal to that given by formula 4.

(a.) We have

$$h < \sqrt{\frac{s}{2a(1-a)}}.$$

In this case the values $\left| \frac{m_i}{s} - p \right|$ are less frequent in the neighborhood of p than they were in (1). The series, however, is typical, because the value p is fit to represent the entire group of the numbers $\left| \frac{m_i}{s} - p \right|$. Lexis explains this case by supposing that we have not to deal with a constant probability, but with one which undergoes chance variations. Events of this type may be compared to drawings from a set of urns which were filled with white and black balls in the following way. A coin is tossed up n times and every time when head appears a white ball is put into the first urn and when tail appears a black ball is put into it. This process is continued until every urn contains n balls, and then from every urn the same number of drawings is made, the ball being replaced after its color is registered.† The

*W. LEXIS, *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, 1877, p. 22. Cfr. E. CZUBER, *Die Entwicklung der Wahrscheinlichkeitstheorie*, *Jahresbericht der Deutschen Mathematiker-Vereinigung* 1898, Vol. 7, p. 85, 233, and the same author, *Wahrscheinlichkeitsrechnung*, 1903, p. 131.

†For another example see E. CZUBER, *Wahrscheinlichkeitsrechnung*, 1903, p. 131.

result of the drawings from one urn depends on the probability of the appearance of a white ball due to the number of black and white balls which it contains, and the total outcome depends on the combination of two chance events. The result of this combination is a distribution according to the $\Phi(\gamma)$ -law.* Events of this type are chance events in the proper sense of the word and the result of one observation does not depend on that of any of the previous observations.

(b.) The differences $\left| \frac{m_i}{s} - p \right|$ are distributed as required by formula (1.) but the constant h is greater than h' computed by formula (4.) In this case the differences are clustered more closely around p than Bernoulli's theorem requires. Series of this type are called typical series with less than normal dispersion. The single results are not independent from each other but there exists a relation between them. According to Lexis such a distribution is due to a law or a norm, which tends to produce a certain numerical value of the relative frequency. Events of this kind are not independent from each other.

(II.) The distribution of the terms $\left| \frac{m_i}{s} - p \right|$ does not follow the $\Phi(\gamma)$ -law. We speak in this case of an irregular distribution† and we call such a series of observations a symptomatic series.‡ These series indicate a change in the system of underlying causes. If the change occurs always in the same direction we may speak of an evolutoric series; if phases of increase alternate with phases of decrease we may speak of an undulatory variation and, finally, if the same changes occur after constant time intervals we may speak of periodic changes or variations.

It follows from this discussion that the interpretation of a sta-

*This is a special case of what Bruns calls the conservation of the $\phi(\gamma)$ -type. See H. BRUNS, *Wahrscheinlichkeitsrechnung und Kollektivmasslehre*, 1906, p. 136.

†L. v. BORTKEWITSCH, *Das Gesetz der kleinen, Zahlen*, 1898. E. CZUBER *Die Entwicklung der Wahrscheinlichkeitstheorie, etc.*, p. 234.

‡W. LEXIS, *Massenerscheinungen*, p. 33. A statement of this terminology may be found also in E. CZUBER, *Wahrscheinlichkeitsrechnung*, p. 322 and E. BLASCHKE, *Vorlesungen über mathematische Statistik*, 1906, p. 22.

tistical number of relative frequency depends to a large extent on the existence of a limit p which the single observations approach and on their distribution around this limit. The chief difficulty of deciding this question in a particular case lies in obtaining a sufficient number of data, because one needs not only a large number of observations, but even a large number of extended series of observations; only in this case would it be possible to compare the actual distribution with that required by formula (1). It is not likely that there exists in psychology an investigation which is based on a number of experiments large enough for this purpose; direct tests of the $\Phi(\gamma)$ -distribution were made until now only in those cases where favorable outside circumstances facilitated the task of collecting material, as e. g. in the statistics of population. Fortunately one may devise methods by which it is at least possible to distinguish between the three cases enumerated under (I). This was done by W. Lexis, L. v. Bortkewitsch and E. Dormoy. Their method is based on the computation of the coefficient of precision by different formulae. The first way of computing the coefficient of precision is indicated by formula (4) which is based on Bernoulli's theorem. Each one of the n series can also be looked at as an observation on the same probability p . Owing to the discrepancies between the n results of these observations it is necessary to combine them by the method of least squares, which is applicable because we have to deal with observations of the same empirical constant. If the number of observations is the same in all these n series, they are equally exact empirical determinations of the probability p . The coefficient of precision is

$$h'' = \sqrt{\frac{n-1}{2 \sum \left(\frac{m_i}{s} - a \right)^2}} = \sqrt{\frac{n-1}{2 [kk]}} \quad (5)$$

and if we have to deal with a typical probability of normal dispersion, it must be equal to the result of formula (4). The ratio of these two values

$$Q = \sqrt{\frac{[kk]s}{(n-1)a(1-a)}} \quad (6)$$

which is called the coefficient of divergence, affords the basis for the decision whether we have to deal with normal dispersion or not. The computation is not as laborious as one might believe.

The answer of the question whether differences in the results of several series of observations are due to chance variations, or whether they indicate a change in the system of underlying causes must be based on the coefficient of divergence. The case (II) will be easiest to recognize by the systematic order of the deviations. Since this case indicates a change of the conditions of the experiment, one may find the law of the variations of the probability by following up in detail the variations of the numbers $\frac{m_i}{s}$ and thus gain an insight into the nature of this change.

A similar interpretation may be given to the case (I, 2b.) The grouping of the numbers $\frac{m_i}{s}$ around p indicates some recognizable influence, which is characterized by a value of the coefficient of divergence smaller than 1. There remain, therefore, only the cases (I. 1.) and (I.2a.) which are similar in that respect that they do not indicate a recognizable law. In the first case we have to deal with a constant group of phenomena and the variations of the observed results are due to the influence of chance. The second case is in so far different from the first as it indicates that the results are due to a group of phenomena which itself is subjected to chance variations.

Our results are not extended enough to compare directly the distribution of the numbers of relative frequency with that required by the probability integral, and we must be satisfied with the decision whether the results show a normal, subnormal or overnormal dispersion. The first step to be taken for this purpose is the computation of the numbers of relative frequency from the numbers of absolute frequency given in the Tables 3-9. These numbers for the "heavier" and for the "lighter" judgments are given in the Tables 10-16 (Appendix pp. 177-180), each subject being given a separate table. The tables are so arranged that the results for every one of the pairs of comparison weights (84, 88, ... 108) are given in a double column under the headings h

and 1 ("heavier" and "lighter"-judgments) in groups of 50 experiments. Numbers standing in one line refer to results obtained in one such group; thus we find, for instance, that for the subject VII the relative frequency of "lighter"-judgments for the comparison weight 88 was 0.020 in the experiments 51-100 of series III. From the numbers of relative frequency one may find the percentages by multiplying by 100. These numbers are empirical determinations of the underlying probabilities of "heavier" and of "lighter"-judgments and one may state the limits of accuracy of such a series of 50 experiments according to Bernoulli's theorem. The usual way of giving the limits of accuracy in a determination by Bernoulli's theorem is to give the probable errors, which is defined by

$$0.476936 \sqrt{\frac{2mn}{s^3}}.$$

This quantity gives the limits of the interval from

$$\frac{m}{s} - 0.476936 \sqrt{\frac{2mn}{s^3}} \text{ to } \frac{m}{s} + 0.476936 \sqrt{\frac{2mn}{s^3}}$$

inside of which the unknown probability p may be expected with the probability $\frac{1}{2}$. The probable errors for $s=50$ are given in Table 17 (Appendix p. 181) under the heading P. E. The first column of this table contains under the heading m the numbers from 1 to 25, and in the next column under the heading n their differences from 50. The probable error in the determination of the probability of an event which was observed k times in a total of 50 experiments may be found in the line which contains the number k no matter whether in the column m or in the column n .

These numbers of relative frequency may be used still in another way. The tables contain data about series of observations each one consisting of 50 experiments in which an event E (giving a "heavier" or a "lighter"-judgment) was observed k times. One may ask, which is the probability of this event in the next experiment. This probability is given by the formula

$$\frac{k+1}{k+(50-k)+2} = \frac{k+1}{52}$$

These values are given for all the possible results of a series of 50 experiments in Table 17 under the headings p' and q' . Values of k which are smaller than 25 must be looked up in the first column under the heading p' , those greater than 25 are given in the column q' . The differences between the values $\frac{k}{50}$ and $\frac{k+1}{52}$ are not great. The corresponding differences for $s=300$

are very small and for this reason a similar table for $s=300$ is not given. The probable errors of a series of 300 experiments may be found by the formula given above; the maximum of the probable error is 0.01947, which is attained for $k=150$.

The numbers at the bottom of the Tables 10-16 in the line marked "Average" are the arithmetical means of the numbers in the same columns and they are, as such, the numbers of relative frequency in the entire series of experiments. These numbers show in a more convincing way the same as the numbers of the smaller series. The probabilities of "heavier" judgments are very small for large negative differences and when the differences decrease these numbers increase at first slowly, but then rapidly and come very close to the unit for somewhat considerable positive differences. With three subjects there are slight breaks in this upward movement, all occurring for large negative differences. These breaks are to be found in the tables for the subjects I, V and VII for the differences 84, and 88; the amount of inversion is small in all the cases. The course of the probabilities of "lighter"-judgments is opposite. These probabilities set in with very high values for large negative differences, decreasing slowly at first, but coming down rapidly to very small values for positive differences.

The numbers given in the Tables 10-16 serve for the computation of the coefficient of divergence. The first business is to find the deviations and the squares of the deviations from the arithmetical mean. After having found the sum of the squares of the deviations we are able to determine the physical precision by formula 5; the values of the combinatoric precision may be found immediately from the data in Tables 10-16 by the application of formula 4. The values of h' and of h'' are given in the

corresponding columns of the Tables 18-24 (Appendix pp. 181-184). Dividing h' by h'' gives the value of the coefficient of divergence which is given in the tables in the columns Q. A glance at these numbers shows that they vary not inconsiderably, and that the range and the amount of variation is very different for the different subjects. The averages of the coefficients of divergence are given here for the different subjects:

Subject	II	1.01
"	VII	1.13
"	I	1.22
"	VI	1.39
"	IV	1.50
"	V	1.62
"	III	1.69

These numbers seem to indicate that we have to deal in the cases of subjects IV, V and III with a slightly overnormal dispersion. It may be doubtful whether subjects I and VI, are cases of normal or slightly overnormal dispersion, but for the subjects II and VII it is almost certain that their result show a normal dispersion. For subject II, for instance, 7 of the Q's are above and 7 are below the unit and if one disregards the result for 104h, which is clearly out of bounds, the results vary between 0.76 and 1.43. With this one may compare the numbers for the coefficients of divergence for the rate of male and female births, which were found by Lexis* in his first series of observations. They are based on the statistics of the birth rates in 34 Prussian counties during 24 months. This gives 34 series of 24 observations each; one series consists of from 639 to 4,766 observations. The values of the Q's vary between 0.72 and 1.47 and 19 of them are beyond and 15 below the unit. The range of variation is therefore a little larger than in our experiments, if one omits the result for 104h, and our results compare very favorably even if one does not

*W. LEXIS, *Das Geschlechtsverhältniss der Geborenen und die Wahrscheinlichkeitsrechnung*, pp. 216-245; *Zur Theorie der Massenerscheinungen*, pp 64.78; the results are reproduced in E. CZUBER, *Wahrscheinlichkeitsrechnung und ihre Anwendung*, 1903, p 324.

omit it. The arithmetical mean of the Q's in Lexis's observations is 1.09, approaching more closely the less satisfactory result for subject VII than that for subject II. The slight deviations from the unit may be well accounted for by the inaccuracy in the determination of h ". Considering the fact that Lexis's results are based on 24 series of observations each one containing at least 13 times as many experiments than ours (the average of the number of "experiments" in one series of Lexis is approximately 2,300 or 46 times as many than in our experiments) one will be justified in believing that the results for the subjects II, VII, I and perhaps also those for subject VI indicate a normal dispersion. The dispersion in the results for the other subjects may be considered to be overnormal. We formulate this result in this way: The statistical numbers of relative frequency which one obtains in psychological experiments show for some subjects a normal dispersion. According to our previous discussion we will conclude that the group of conditions on which the formation of a judgment depends may remain fairly constant for certain subjects. The answer of the question whether or not we have to deal with a typical probability cannot be undertaken on the basis of our present material; it must be postponed until further results are at hand.

The demonstration of the normal dispersion of the results is interesting in several respects. At first, the number of examples in which a normal dispersion could be demonstrated is not great. Besides this *pretium raritatis* there comes in the practical conclusion that experimental psychology is certainly capable of becoming as much of a science as any other branch of knowledge which takes its material from statistical results. The fact that psychology has to deal with probabilities of normal dispersion gives this science a distinguishing feature of which not many statistical sciences can boast at present. The normal dispersion of the numbers of relative frequency also proves that the instability of mental states is not so very great as some writers were inclined to believe. For some time the view was very popular among certain philosophers, that a mental state may be connected with very widely different physical conditions, and that the same groups of physical conditions may be connected with different

mental states. Our results show that at least the latter part of this statement is not quite correct, for in some subjects the results are such as they would be, if they were chance phenomena under the influence of a constant group of conditions. Training in psychological experiments seems to have the tendency of producing in a subject the ability of reacting under certain influences, and we may define a subject with training in psychological experiments as a subject, which is capable of reacting in such a way that the reactions are due to a constant group of conditions which are so far under the control of the subject that they may be varied at the will of the experimenter. Subject II who has the smallest coefficient of divergence, had several years experience in psychological experiments. Subjects VII and I had considerable ability for this work, whereas subject III was the man who could not be trained to the proper use of the "guess"-judgments, and subject V was a young lady with little practice in experimental work.

The observed values of the coefficients of divergence prove that there is no appreciable change in the results which are taken at different times for at least some subjects. The next question is whether there do not occur any changes within a series. Besides the variability of the conditions of a single lifting (e. g. variations in the velocity or height of lifting) there is one circumstance which might be of influence on the results: Namely the taking of the experiments in groups of 35. It takes about three minutes to perform such a series and one may ask whether this does not produce an influence (fatigue, strain of attention,) which puts the judgments in the latter part of one series under conditions which are entirely or partially different from those in the preceding part of the series. One may try to settle this question by means of the introspective evidence that there was or that there was not an appreciable influence of fatigue or lack of attention. It is very likely that this question refers to conditions which are not favorable to introspection, and we take the attitude that an objection raised against a series of experiments on the ground of introspective evidence makes the series suspect, but that the absence of any such objection based on introspection does not put the result beyond doubt. The introspective evidence, as a matter of fact, forms only one part

of the outcome of the experiments. The objective records are the other part, and it would be one-sided to neglect one part for the sake of the other. This attitude is similar to that which one takes towards sets of measurements which are the results of observations of a physical quantity, where an opinion about the value of the observations formed in collecting the data must be verified by a minute examination of the results in order to establish the objective value of the results. There were in our experiments no spontaneous complaints from the part of the subjects as to the length of the experiments and cautious questioning never revealed any discomfort caused by the protracted experimentation. Introspective evidence against our experiments is therefore not at hand.

A change in the conditions, which is not noticed by introspection, must be expressed by a variation of the results i. e. in our case by a different frequency of the "heavier", "lighter" and "guess"-judgments in the different parts of one series of 35 experiments. We shall make this investigation on the numbers for the combined h and hg judgments. It happens that the records of the single series are naturally divided in five groups, each one containing the judgments given during one turn of the table, i. e. one judgment for every pair of comparison weights. The number of "heavier"-judgments given during one turn of the table can be found by counting these judgments in one column of the records. Dividing the number of "heavier"-judgments in the 1, 2, ..., 5 column by the total number of judgments of a column one finds the relative frequency of these judgments during the first, second, ..., fifth part of a series. These five numbers of relative frequency are, for one individual, only slightly different, as may be seen from Table 25 (Appendix p. 185). The numerals at the head of the columns (1-5) refer to the five turns of the table in one series and the numbers in one column give the relative frequency of the "heavier"-judgments during the first, second, ..., fifth turn of the table for the different subjects. The variations of the numbers referring to one person are small and they are not systematic, increase and decrease interchanging without regularity.

The total number of judgments given during one particular turn of the table was for the subjects I, II and III, 840 and for

- the other subjects 560. The numbers in one line of Table 25 give the ratios $\frac{m_i}{s}$ ($i = 1, 2, 3, 4, 5$), where m_i is the number of "heavier"-judgments in the column i . Let us call the arithmetical mean of these numbers (referring to one subject) p and designate by $\sum \left(\frac{m_i}{s} - p \right)^2$ the sum of the squares of the deviations of the numbers $\frac{m_i}{s}$ from their mean. We define furthermore, by q the probability of a judgment which is not a "heavier" judgment so that $q = 1 - p$, and we designate by u the mean error of the arithmetical mean of the numbers $\frac{m_i}{s}$

$$u = \sqrt{\frac{1}{4.5} \sum \left(\frac{m_i}{s} - p \right)^2}$$

These numbers may be found from the data of the preceding table and are given under the corresponding headings of Table 27. They serve for the determination of the coefficient of divergence Q , which in problems of this kind is found by the formula

$$Q = \sqrt{1 + \frac{s-1}{5pq} \sum \left(\frac{m_i}{s} - p \right)^2}. \quad (8)$$

We determine, furthermore, a quantity M which is defined by

$$M = \sqrt{\frac{s-1}{5s} \sum \left(\frac{m_i}{s} - p \right)^2}. \quad (9)$$

This quantity is called the component of physical variation* because it indicates the amount of variations of the underlying

*This term (Physische Schwankungskomponente) is due to W. Lexis, *Über die Theorie der Stabilität statistischer Reihen*, *Jahrbuch f. National Ökonomie u. Statistik*, Vol. 32, 1879. L. v. BORTKEWITSCH, *Das Gesetz der kleinen Zahlen*, 1898, p. 30, calls the quantity M "absoluter Fehlerexcedent." Cfr. E. CZUBER, *Wahrscheinlichkeitsrechnung*, 1903, pp. 317-321.

complex of causes which determines the probability of the event in question in the different parts of the series. M and Q are in a simple relation shown by the formula

$$M = u' \sqrt{Q^2 - 1}.$$

where the factor u' represents the quantity

$$u' = \sqrt{\frac{pq}{s}}$$

These relations may be used for checking the computation by the first formula for M .

We will illustrate the course of the computation by working out the results for the first subject. The total number of judgments in each one of the five columns was 840, and the numbers of "heavier" and "heavier guess" judgments are given in the following little table under the heading m_i . By dividing these numbers by 840 one obtains the relative frequencies of these judgments in the five columns; the arithmetical mean of these numbers is found to be $p = 0.5374$. The column under the head-

ing $\frac{m_i}{840} - p$ gives the deviations from the arithmetical mean.

The sum of the positive deviations is 0.0449 and the sum of the negative deviations is 0.0450; this difference is due to the abbreviation of the result. The last column of the table gives the squares of the deviations, the sum of which is found to be 0.00202773.

m_i	$\frac{m_i}{840}$	$\frac{m_i}{840} - p$	$\left(\frac{m_i}{840} - p\right)^2$
472	0.5619	+0.0245	0.00060025
461	0.5488	+0.0114	0.00012996
444	0.5250	-0.0124	0.00015376
459	0.5464	+0.0090	0.00008100
424	0.5048	-0.0326	0.00106276
$\Sigma \frac{m_i}{840} = 2.6869$		$\Sigma \left(\frac{m_i}{840} - p\right)^2 = 0.00202773$	
$p = 0.5374$		$\frac{1}{5} \Sigma \left(\frac{m_i}{840} - p\right)^2 = 0.00040555$	

Dividing the sum of the squares of the deviations by $5.4=20$ and taking the square root one finds the number $u = \sqrt{0.0001039} = 0.0101$, the mean error of the arithmetical mean. Now one proceeds to find M by means of formula 9.

$$\begin{array}{rcl}
 \log 839 & = & 2.92376 \\
 \log 840 & = & 2.92428 \\
 \hline
 \text{Difference} & = & -0.00052 \\
 \log 0.00040555 & = & 0.60804-4 \\
 \hline
 \text{Sum (regarding} & & \\
 \text{the sign)} & = & 0.60752-4 \\
 \hline
 \text{Dividing by 2} & = & 0.30376-2 = \log M \\
 M & = & 0.020126
 \end{array}$$

The computation of the quantity Q by formula (8.) and the check of the computation of M may be effected in this way.

$$\begin{array}{rcl}
 \log 0.00040555 & = & 0.60804-4 \\
 \log 839 & = & 2.92376 \\
 \hline
 \text{Sum} & = & 0.53180-1 \\
 \log pq & = & 0.39551-1 \\
 \hline
 \text{Difference} & = & 0.13629 = \log (Q^2-1) \\
 Q = \sqrt{2.1369} & = & 1.462 \\
 \log pq & = & 0.39551-1 \\
 \log 840 & = & 2.92428 \\
 \hline
 \text{Difference} & = & 0.47123-4 \\
 \hline
 \text{Dividing by 2} & = & 0.23562-2 \\
 \log \sqrt{Q^2-1} & = & 0.06814 \\
 \hline
 \text{Sum} & = & 0.30376-2 = \log M, \text{ as above.}
 \end{array}$$

In Table 26 (Appendix p. 185) data are given which facilitate the control of the computation; the denotations used in this table are the same as those used in the text. The final results for the coefficients of divergence and for the components of physical variation are given in Table 27, (Appendix p. 185) under the corresponding headings. It must be kept in mind that in problems like this, where the coefficient of divergence is computed by formula (8), the result must necessarily be a value greater than the unit, and one can require only that it does not exceed this value too much. The value $Q = \sqrt{2} = 1.414$ is the limit which ought not to be exceeded, because in this case the physical variation is equal to the accidental variation. Only in two of our subjects do the Q 's exceed this limit (I and VI) the margin being small (0.048 and 0.164.) The average of all the Q 's is 1.345 which comes very near the case of equality of physical and accidental variation the difference being on the safe side. It will be noticed that for the subjects II and VII, for whom we have found above a normal dispersion, the values of Q come nearest to 1.414, the differences being 0.006 and 0.018. From this it follows that there is no difference between the results of experiments made at different parts of a series of 35.

CHAPTER III.

ON THE METHOD OF JUST PERCEPTIBLE DIFFERENCES.*

The method of just perceptible differences is the oldest of all psychophysical methods. After attention was called to the fact that our judgments on the equality of two stimuli are not exact in any strict sense of the word, it was obviously possible to investigate the limits within which two stimuli may vary without a difference being perceived. This was indeed the problem of Lambert† and Bouguer‡, the first investigators who tried to determine this range of objective variability with subjective equality for optical sensations. When Lambert lighted a screen by a candle placed in front of the centre and tried to find the range inside of which there was no appreciable difference in the intensity of the light, he had in his result a quantity essentially identical with the threshold of difference as determined by the method of just perceptible differences. This method came more into prominence by the experiments of Weber and, later on, of Fechner, who used the notion of a just perceptible difference for his theory of the measurement of sensations. The notion of the threshold became identical with that of the just perceptible difference, which was defined as the smallest difference which could be perceived. The fact that smaller differences may be perceived and that larger differences may not be detected was explained by accidental errors. The notion of the just perceptible difference thus became the fundament of quantitative psychology and this method is used directly or indirectly in most of the investigations which deal with Weber's Law or with the problem

*A short abstract of this chapter was given in the *Psychological Review*, Vol. 14, July, 1907, pp. 244-253.

†LAMBERT, *Photometria, sive de mensura et gradibus luminis, colorum et umbrae*, 1760, in OSTWALD's *Klassiker d. ex. Naturwiss.*, 31-33.

‡BOUGUER, *Traité d'optique sur la gradation de la lumière*, 1760.

of the accuracy of sense perception, in spite of the fact that there is at present a tendency to substitute one of the error methods for the method of just perceptible differences in accurate work.

The form in which the method of just perceptible differences was used by the first investigators was open to many criticisms and it lasted some time until this method obtained the form in which it is used to-day. The merit of this perfection is chiefly due to Fechner, who also gave the name to this method. In determining the threshold by the method of just perceptible differences* one starts from two stimuli which are equal and, keeping one stimulus constant, one increases the other until a difference is perceived. This difference is put down as the result of an observation on the just perceptible positive difference. The second step consists in finding the just imperceptible positive difference. This is done by allowing a supraliminal difference to decrease until it ceases to be perceptible. The largest difference which is not noticed is put down as an observation of the just imperceptible positive difference. The just perceptible and the just imperceptible positive differences are, generally speaking, not identical. In a similar way one finds the just perceptible and the just imperceptible negative differences. In an investigation which aims at some accuracy it is indispensable to make a considerable number of experiments in order to eliminate the errors of observation which express themselves in differences of the numerical results. It seems to be justified to treat the results by the algorithm of the method of least squares, since one considers the just perceptible difference a physical quantity the observations of which are subjected to chance errors. The probable error of the final result may serve as an indication of the greater or smaller variability of the psychophysical conditions. The arithmetical mean of the just percep-

*For the description of this method see: FECHNER, *Über die psychischen Massprincipien*, *Phil. Stud.* Vol. IV, 1888, p. 161 sqs. besides the "Elemente" Vol. I and II and the "Revision"; WUNDT, *Das Webersche Gesetz und die Methode der Minimaländerungen*, *Phil. Stud.* Vol. I, pp. 556 sqs. *Physiologische Psychologie*, 5 ed., Vol. I, pp. 470 sq., 475-479; TITCHENER, *Experimental Psychology*, 1905, Vol. II, Part I, pp. 55-69, and Part 2, pp. 99-143 with historical notes.

tible positive and of the just imperceptible positive difference defines the threshold in the direction of increase, and the arithmetical mean of the just perceptible and of the just imperceptible negative difference gives the threshold in the direction of decrease.*

The practical application of this method meets with two very peculiar difficulties which impair its serviceability and necessitate a change in the experimental procedure. It seems that it is essential for the result, by which steps the threshold is approached. After the first rough determination is made, one tries to get a more accurate one by using smaller intermediate steps. In experimenting with this new series one notices very soon that the subject is more apt to perceive smaller differences in the determination of the just perceptible difference, and not to perceive larger differences in the determination of the just imperceptible difference than he was before. Usually one explains this fact by the influence of expectation, because the subject knows that an imperceptible difference is to be increased and a perceptible difference is to be diminished. In order to avoid this inconvenience one has employed two means. The first was to use only trained subjects who are free from the influence of expectation, and the second consisted in applying the stimuli in irregular order. The first way cannot always be used, because one frequently is obliged to experiment on untrained or half-trained subjects, and this requirement of training on the part of the subject, furthermore, disposes of the possibility of using this method for practical purposes. The determination whether the sensitivity of a patient is below or above the normal must necessarily be made on untrained subjects, and until now no other method but the method of just perceptible differences could be used for this purpose. The method of presenting the comparison stimuli in irregular order has the inconvenience that the results cannot be worked out by the algorithm of the method

*The threshold in the direction of increase and that in the direction of decrease are frequently combined in order to obtain a generalized threshold or to eliminate constant errors. This procedure, which was first suggested by VOLKMANN, gives a result which seemed to be not entirely clear in its signification; see TITCHNER, *l. c.* Part 2, p. 112 sqs.

of just perceptible differences. This modification of the method of just perceptible differences is called the method of irregular variation. The results obtained in this way are usually worked out by an algorithm similar to that of the error methods.*

The second difficulty is due to the fact that frequently a difference is not noticed after a smaller one has been perceived in a series for the determination of the just perceptible difference, or that a difference is noticed after a larger difference has been imperceptible in a determination of the just imperceptible difference.† The question arises, what must be done with such results? Some investigators take the view that these series must be ruled out and that only those series must be kept for the final computation, where no such inversion occurs and where the two classes of judgments are strictly separated. Another possibility consists in not going beyond the first difference which is perceived or which fails to be perceived, thus avoiding these dubious cases. It is obvious of course that one may try to escape this difficulty by taking very large intermediate steps, but usually one is afraid of doing this, because one seems to renounce the hope of obtaining a determination of the threshold the accuracy of which may compare favorably with that of determinations by one of the error methods.

To these practical difficulties comes the theoretical problem of finding the relation between the results of the method of just perceptible differences and those of the error methods, especially of the method of right and wrong cases. This question is a very urgent one since the individual experiments are the same for

*WUNDT, *Physiologische Psychologie*, 5 ed., 1902, Vol. I, p. 478. Some criticism of Wundt's view on the relation of the method of just perceptible differences to the method of irregular variation may be found in E. B. HOLT, *Classification of Psychophysical Methods*, *Psych. Review*, Vol. XI, Nov. 1904, p. 348, who contends that the method of irregular variation does not approach the error methods, as Wundt says, but that it is identical with them. He furthermore remarks that the method of just perceptible differences is not a method, if this word is used in its proper sense, but the statement of the intention to find a significant value for the threshold.

†Several examples of such series were published lately by WILHELM SPRECHT, *Das Verhalten von Unterschiedsschwelle und Reizschwelle im Gebiete des Gehörsinnes*, *Archiv. f. d. ges. Psychologie*, Vol. 9, 1907, p. 207.

both methods. The difference merely consists in using several pairs of comparison stimuli in a particular order in the method of just perceptible differences, or in any order in the method of irregular variation, and only one pair in the method of right and wrong cases. To this comes that the results of the error methods, as well as those of the method of just perceptible differences seem to confirm Weber's Law, in spite the fact that they are not comparable among each other. Some investigators believed that the results of the method of right and wrong cases were in no direct relation to those of the method of just perceptible differences, but others tried to establish such a relation by mathematical formulae; this relation however proved to be of very complicated nature. Of course one might take the view that one of these methods is not legitimate, but this view, which is somewhat narrow, would be justifiable only if the interpretation of the results of either one of these methods were absolutely clear. The difficulties of the method of just perceptible differences, which were mentioned above, are considered strong arguments against the use of this method and at present many, if not most psychologists would favor the error methods against the method of just perceptible differences, if they had to choose between them, in spite the fact that the foundations of the method of right and wrong cases are little known and not very well understood.

For the development of the psychophysical methods one circumstance proved to be of fundamental importance: The introduction of the theory of errors by Möbius and Fechner, which necessitates the division of the judgments into the two classes of correct and wrong cases. The theory of errors of observation gave some insight into the nature of the error methods and one could hope to find the relation of these methods to the method of just perceptible differences, because the Gaussian coefficient of precision (the "*mensura precisionis*") seemed to give a measure of the accuracy of sensations similar to that afforded by the smallest perceptible difference. It escaped attention for a long time that the application of the theory of errors of observation, though helpful for certain purposes, entirely excludes the notion of a just perceptible difference. It

is the merit of Jastrow and Cattell to have called attention to this fact. The theory, indeed, starts from the supposition that the probability of every error is a function of the size of this error; the theory makes certain assumptions as to the nature of this dependence, which imply that no error, no matter how large, is impossible although its probability may be very small. One also finds that the greatest error which is likely to be committed in a certain series of observations depends on the number of observations, so that the more extended the series is the greater the largest error becomes which is likely to be committed. Cattell, supposing that the errors in judgments on differences of intensity of two stimuli follow the same law as the errors of observation,* concludes that there does not exist a just perceptible difference in any absolute sense of the term, because the smallest difference beyond which all the judgments of a series are correct depends on the number of observations in this series. The supposition that there exists a difference which is always perceived is, therefore, in contradiction with the fundamental supposition of the method of right and wrong cases. Cattell stands in his demonstration entirely on the ground of the theory of errors of observation and he does not go beyond it. His arguments prevail against the criticisms of his view, some of which miss entirely the point of his argument, e. g. the experimental demonstration that one may make the difference between two light intensities so small that it cannot be perceived.

It seems that the source of these difficulties is the introduction of the distinction between correct and wrong judgments. This is a logical category. What is immediately given are not right and wrong judgments but the judgments "greater", "equal" and "smaller"; the correctness or incorrectness of the judgments is a secondary feature. By introducing this distinction between correct and incorrect judgments it becomes necessary to dispose in some way of the equality cases, which as a matter of fact were so troublesome a feature in the method of right and wrong cases, that some investigators have tried to get rid of them by not allowing the subject to pass the judgment "equal." The imme-

*G. S. FULLERTON and J. McKEEN CATTELL, *On the Perception of Small Differences*, 1892, p. 12 sqs.

diate data of experiments for the determination of the sensitivity of a subject are the observed frequencies of the cases in which the judgments "greater", "equal" and "smaller" were passed. We will try to base our judgment on the sensitivity of a subject on these percentages for various differences, without obliterating one feature of the results by eliminating a class of judgments, and without introducing the logical category of right and wrong judgments. We shall use for this purpose the notion of the probability of a judgment of certain type, which was introduced in the preceding chapters.

Let us suppose a subject compares n pairs of stimuli which have one stimulus, the standard, in common. Let the order in which the comparison stimuli are presented be the same for all the pairs, e. g. let the standard be the first stimulus of every pair. Let us call the different stimuli with which the standard is compared $r_1, r_2, \dots, r_n$ and let us suppose that the pairs are presented in such an order that

$$r_1 < r_2 < \dots < r_n.$$

Keeping all the conditions which might possibly influence the judgment as constant as possible we give this series repeatedly to a subject. The judgments will vary between "greater," "lighter" and "equal" and there exists under the conditions of the experiments a definite probability for every comparison resulting in a judgment of one of the three types. The probabilities that a "greater"-judgment will be given may be called

$$p_1, p_2, \dots, p_n$$

where p_k is the probability that the judgment "greater" will be given on the comparison of the standard with the comparison stimulus of the k^{th} pair. The probabilities that a "greater"-judgment will not be given are correspondingly

$$\begin{aligned} 1-p_1 &= q_1 \\ 1-p_2 &= q_2 \\ &\dots\dots\dots \\ 1-p_n &= q_n. \end{aligned}$$

The cases where a "greater"-judgment is not given comprise those cases where the comparison stimulus was judged equal to the standard, and those cases where it seemed to be smaller than the standard. The comparison stimuli may be chosen in such a way that the probability of a "greater"-judgment is small for the stimuli at the beginning of the series, and large (close to the unit) for the stimuli at the end of it. Presenting this series to the subject we will obtain a determination of the just perceptible positive difference in the comparison stimulus of the first pair on which the judgment "greater" is given, all the preceding pairs being judged "smaller" or "equal." The probability that the k^{th} pair is the first to be judged "greater" is the compound probability that the comparison stimulus of this pair is judged "greater" and that the judgment "greater" is not passed on any one of the pairs with smaller comparison stimuli. The probability, therefore, that r_k will be obtained as a determination of the just perceptible difference is given by

$$P_k = q_1 q_2 \dots q_{k-1} p_k = p_k \prod_{i=1}^{i=k-1} q_i \dots \dots \dots (I.)$$

This quantity is different for every pair. In a series in which the probabilities of "greater"-judgments increase from very small values at the beginning of the series to large values, the P 's increase at first and decrease after having attained a maximum, if certain conditions are fulfilled which will be enumerated later. The results of N determinations of the just perceptible positive difference after being brought in proper order will have this form:

The stimulus r_1 was the first to be judged "greater" N_1 times,

" " r_2 " " " " " " N_2 "

.....

The stimulus r_n was the first to be judged "greater" N_n times, where

$$N_1 + N_2 + \dots N_n = N.$$

The algorithm of the method of just perceptible differences prescribes to take the arithmetical mean of all these determinations.

for the final determination of the just perceptible positive difference. This average is given by

$$\frac{1}{N}(r_1 N_1 + r_2 N_2 + \dots + r_n N_n).$$

In a considerable number of determinations every stimulus r_k will tend to occur in a number of times proportional to its probability P_k . The most probable result for a great number of determinations of the just perceptible positive difference is therefore

$$T = r_1 P_1 + r_2 P_2 + \dots + r_n P_n. \quad \dots\dots\dots (II.)$$

This relation shows that the result of the method of just perceptible differences depends on the probabilities of the judgments of different types and that, therefore, its basis is identical with that of the error methods. The second remark which we have to make regards equation I. The value of P does not change if the order of the terms $q_1, q_2, \dots, q_{k-1}, p_k$ is changed, because a product is independent of the order of its factors. It does not matter either if a stimulus r_{k+1} is given before r_k , because p_{k+1} does not enter into relation I. This means that the probability of a stimulus being the smallest on which a "greater"-judgment is given does not depend on the order in which the pairs are presented. The method of just perceptible differences, therefore, is not tied down to the rule of presenting the stimuli in the order of their magnitude.

Two interpretations may be given to relation II. The first is based on the notion of the mathematical expectation; its psychological bearing is less obvious but it is more serviceable for certain practical purposes. Multiplying each stimulus with the probability that it will be obtained as a result of the method of just perceptible differences gives what is called the mathematical expectation for this stimulus being the threshold, and the sum of these products for all the pairs of the series gives the mathematical expectation for the entire series. The mathematical expectation for N repetitions of the same event is N times that for a single event and taking the results of many observations for its determination gives the result a greater exactitude. This interpretation of the algorithm of the method of just percep-

tible differences is independent from any particular hypothesis as to the law of distribution. One may try to make its meaning clear in this way. Let us suppose that A pays to B a sum of money proportional to the intensity of the comparison stimulus which is obtained as a determination of the just perceptible positive difference, how much must B pay to A in order to induce him to make this agreement and to make it a fair wager? Relation II gives the answer that B's payment must be proportional to T, because in this case the expectation of A is equal to that of B.

The second interpretation is based on the signification of the arithmetical mean for symmetrical distributions. Taking the average of a series of observations means that one tries to determine the most probable value, if the distribution is symmetrical. The most probable result of one determination by the series $r_1, r_2, \dots, r_n$ is the one for which the product of formula I is a maximum. P_k is a maximum if

$$P_{k-1} < P_k > P_{k+1}$$

or introducing the corresponding expressions

$$p_{k-1} \prod_{i=1}^{i=k-2} q_i < p_k \prod_{i=1}^{i=k-1} q_i > p_{k+1} \prod_{i=1}^{i=k} q_i \dots (III.)$$

By splitting up this relation into two and eliminating the common factors we obtain the conditions

$$p_{k-1} < q_{k-1} p_k \quad \text{and} \quad p_k > q_k p_{k+1}$$

which are identical with

$$\frac{p_{k-1}}{1-p_{k-1}} < p_k \quad \text{and} \quad \frac{p_k}{1-p_k} > p_{k+1} \dots (IV.)$$

It is, furthermore, a fact that the probability of a "heavier"-judgment becomes the greater the smaller negative differences, and the greater positive differences of the stimuli become, so that relation IV must be simultaneous with

$$p_{k-1} < p_k < p_{k+1} \dots (IVa.)$$

This relation shows that the position of the maximum of P_k depends on the probability of a "heavier"-judgment for this

pair and on those for the pairs immediately preceding and immediately following. The formulae IV and IVa give the conditions, that the value of a certain P_k is greater than those in its neighborhood. If these conditions are fulfilled it does not follow that P_k is the absolute maximum (i. e. greater than any other value of the series), but if P_k is the absolute maximum the conditions IV and IVa must be fulfilled. Let us suppose that the first k stimuli $r_1, r_2, \dots, r_k$ are chosen in such a way that

$$P_1 < P_2 < \dots < P_k$$

and let us consider the possible effect of our choice of the next stimuli. Does it depend on our choice of the stimuli $r_{k+1}, r_{k+2}, \dots, r_n$ that P_k is the maximum, or is the peculiarity of r_k having the greatest probability of being observed as the just perceptible difference based on some particular quality of this stimulus? In the latter case we will have in our result a determination of the just perceptible difference irrespectively of the way in which we approach it, but in the first case this quantity would be different for different series of comparison stimuli. We suppose that the probabilities of "greater"-judgments are analytic functions of the intensities of the comparison stimuli, so that we have

$$\begin{aligned} p_1 &= \Psi(r_1) \\ p_2 &= \Psi(r_2) \\ &\dots\dots\dots \\ p_n &= \Psi(r_n) \end{aligned}$$

and we choose the stimulus r_{k+1} only slightly different from r_k , so that the higher powers of the difference

$$r_{k+1} - r_k = \delta$$

can be neglected. The second part of relation IV has the form

$$\frac{\Psi(r_k)}{1 - \Psi(r_k)} > \Psi(r_k + \delta).$$

The value of $\Psi(x)$ is, by its nature of being a mathematical probability, smaller than 1, and the term on the left side of the

above relation is the sum of a geometric series. Developing $\Psi(r_k + \delta)$ in a power series we find

$$\Psi(r_k) + \frac{\delta}{1!} \Psi'(r_k) + \frac{\delta^2}{2!} \Psi''(r_k) + \dots < \Psi(r_k) + \Psi(r_k)^2 + \Psi(r_k)^3 + \dots$$

or

$$\delta \Psi'(r_k) + \frac{\delta^2}{2!} \Psi''(r_k) + \dots < \Psi(r_k)^2 + \Psi(r_k)^3 + \dots$$

Neglecting the powers of δ and summing up the series we find

$$\delta < \frac{\Psi(r_k)^2}{[1 - \Psi(r_k)] \Psi'(r_k)} \dots \dots \dots (V.)$$

as the condition with which we must comply in our choice of the stimulus r_{k+1} in order to make P_{k+1} smaller than P_k . The terms $\Psi(r_k)$ and $1 - \Psi(r_k)$ are necessarily positive numbers and $\Psi'(r_k)$ is also positive, because $\Psi(r)$ is an increasing monoton function; δ is, therefore, positive. From this it follows that it is always possible to choose a stimulus r_{k+1} greater than r_k in such a way that r_k is more likely to come out as the result of the determination of the just perceptible difference than r_{k+1} . This may be done by choosing r_{k+1} only slightly different from r_k .

It is obvious that we may confine our considerations of the conditions for $P_{k+1} > P_k$ to the case $\Psi(r_k) < \frac{1}{2}$, because P_{k+1} must necessarily be smaller than P_k if $\Psi(r_k)$ is greater than $\frac{1}{2}$. We put

$$\Psi(r_k) = \frac{1}{2} - e$$

and find that P_{k+1} will be greater than P_k if

$$[1 - \Psi(r_k)] \Psi(r_{k+1}) > \Psi(r_k)$$

or

$$(\frac{1}{2} + e) \Psi(r_{k+1}) > (\frac{1}{2} - e)$$

which determines the relation

$$\Psi(r_{k+1}) > \frac{1 - 2e}{1 + 2e} \dots \dots \dots (V.)$$

This fraction is smaller than 1 and it may represent, therefore, a mathematical probability. It remains to show that it may

represent the probability of a "greater"-judgment on one of the stimuli which may be chosen as the $k+1$ pair of our series. These stimuli have to satisfy the relation

$$\Psi(r_k) < \Psi(r_{k+1}).$$

We find for the greatest value of the difference $\Psi(r_k) - \Psi(r_{k+1})$

$$\frac{1}{2} - e - \frac{1-2e}{1+2e} = -\frac{(1-2e)^2}{2(1+2e)}$$

which is always negative. From this it follows that it is always possible to find a stimulus $r_{k+1} > r_k$ which satisfies the relation V. as long as $p < \frac{1}{2}$. The stimulus determined in this way is more likely to be observed as a result of a determination of the threshold by the method of just perceptible differences than the preceding stimulus.

These considerations show the importance of the choice of the comparison stimuli. The generality of our discussion is not impaired by the supposition that there is only one maximum of the P's, because if there are several values which satisfy relation IV we will have to consider an intermediate value (the mean) and the influence of our choice of the comparison stimuli is equal to the sum of the disturbances in the position of the single maxima. The practical bearing of this part of the demonstration is that one must not pick out the comparison stimuli at random, but one will choose them according to a principle. The most obvious rule will be to choose equidistant values

$$\begin{aligned} r_1 &= r_1 \\ r_2 &= r_1 + d \\ r_3 &= r_1 + 2d \\ &\dots\dots\dots \\ r_n &= r_1 + (n-1)d. \end{aligned}$$

The choice of d will depend on the accuracy aimed at in the determination of the threshold.

The outcome of a definite series of experiments for the determination of the just perceptible difference is a well defined quantity, but whether the final average has the signification of a most probable result depends on the distribution of the observed values.

The discussion of this question will show that this supposition may be made under certain conditions. The result of the method of just perceptible differences, therefore, may have a significance independent from the particular series of experiments by which it is reached.

The arithmetical mean is the most probable value of a set of observations, if the distribution is symmetrical i.e. if positive and negative deviations have the same probability. The $\Phi(\gamma)$ -law is only a special case of symmetrical distributions. Under what conditions will the values $r_k N_k$ be distributed symmetrically around their mean? The nature of this distribution obviously depends on the values of the P's. The P's are constituted of the p's and of the q's of the pairs and since it depends on our choice which pairs we will use, no a priori statement is possible in regard to any particular series unless one knows the pairs and the respective probabilities of "h" judgments. Generally the distribution will not be symmetrical, but it may very well be that a particular series has a symmetrical distribution or one which approaches this type. A priori one even cannot make the supposition that the distribution is regular i.e. that it shows an uninterrupted increase at first and, after having attained a maximum, an uninterrupted decrease. Indeed, if the comparison stimuli are picked out entirely at random it may very well be, that the P's increase in value at first, then after having reached a secondary maximum decrease and increase again later on. This will be the case if there are in the series two or more comparison stimuli with only slightly different probabilities, which are smaller than $\frac{1}{2}$, for the appearance of a "greater"-judgment. By taking the stimuli in equal intervals one is to some extent guarded against this eventuality, but if one was unfortunate in the choice of the comparison stimuli one cannot eliminate this influence by any amount of care in the performance of the experiments, or by the combination of any number of observations. It is not possible either to eliminate the influence of a skew distribution by taking the arithmetical mean of a great number of experiments. On account of the fact that one cannot make any statement about the distribution, it is impossible to say whether the final result of an individual series of

experiments by the method of just perceptible differences has the character of being the most probable value of the threshold or not.

The aspect of the problem is entirely different if one has to deal not with the results of one series of comparison stimuli, but with the results of several series with different arrangements of the pairs of comparison stimuli. Owing to the beautiful theorem which Bruns calls the conservation of the $\Phi(\gamma)$ -type this mixture of independent distributions tends to produce the $\Phi(\gamma)$ -distribution. We have therefore to deal in this case with a symmetrical distribution and the arithmetical mean is the most probable value. In the apparently trivial caution "not to approach the threshold always by the same steps" lies to a large extent the justification of the algorithm of the method of just perceptible differences. Most investigators, as a matter of fact, used different series of comparison stimuli for their determinations of the threshold, and this mixture of distributions produced a symmetrical distribution which gave the results a signification independent of the comparison stimuli which had been used. One might have surmised that the results of the method of just perceptible differences must have a definite signification because the results of different investigators are in general agreement in spite of large individual differences. It is hardly imaginable that a regularity like Weber's Law could have been found and verified by independent observations, if every single investigator had determined another quantity, as would have been the case if the results depended essentially on the choice of the comparison stimuli. A very important practical caution for the application of the method of just perceptible differences follows from these considerations: Do not use always the same series of comparison stimuli, but use as many different arrangements as can be done conveniently. In an extended series of observations one will go about systematically in choosing the arrangements. In agreement with the procedure suggested above one will use a series of stimuli of the same difference, but instead of beginning with r_1 one will use a stimulus slightly different, say r'_1 , so that the second arrangement will be

$$\begin{aligned}
 r'_1 &= r_1 \\
 r'_2 &= r'_1 + d \\
 r'_3 &= r'_1 + 2d \\
 &\dots\dots\dots \\
 r'_n &= r'_1 + (n-1)d
 \end{aligned}$$

If one intends to use k different arrangements one will choose the stimuli so that no point of the interval is favored which can be done by making $r'_1 - r_1 = \frac{d}{k}$.

If we have succeeded in choosing d in such a way that none of the pairs complies with relation V (which can be done by choosing d not too small) the P 's will increase at first up to a certain maximum, which is reached for the stimulus r_k , and the value of r for which $p = \frac{1}{2}$ must be in one of the intervals $r_{k-1} r_k$ or $r_k r_{k+1}$. If we have chosen our comparison stimuli in such a way as not to favor any value, there will be an even probability that the stimulus for which P is a maximum is greater or smaller than the stimulus for which $p = \frac{1}{2}$. From this it follows that in a large number of determinations of the just perceptible positive difference made with different series of comparison stimuli, the most probable result is given by that amount of difference for which there exists the probability $\frac{1}{2}$ that the judgment "greater" will be given.

Relation IV explains the surprising fact that the just perceptible difference seems to become smaller, if new comparison stimuli are interpolated. The result of such an interpolation may be as follows; As a rule this interpolation will be made systematically and one will not only not avoid interpolating new pairs in the neighborhood of the point where one expects to find the threshold, but one will be careful to do so since this is the region in which one is most interested. This interpolation of new pairs may produce a shifting of the most probable value, which will not be the same in the new series as it was in the first. In regard to this change the following rule holds: If the most probable result varies it must become smaller necessarily. Indeed, let there be three stimuli r_{k-1} , r_k and r_{k+1} for which the probabilities of "greater"-judgments are p_{k-1} , p_k and p_{k+1} . The probabilities that one of these stimuli will be obtained as a deter-

mination of the just perceptible difference are P_{k-1} , P_k and P_{k+1} respectively. If there exists the relation

$$P_{k-1} < P_k > P_{k+1}$$

we also must have

$$\frac{P_{k-1}}{1-P_{k-1}} < P_k \text{ and } \frac{P_k}{1-P_k} > P_{k+1}.$$

Now let us interpolate a new stimulus r'_k between r_{k-1} and r_k and another stimulus r'_{k+1} between r_k and r_{k+1} so that

$$\begin{aligned} r_{k-1} < r'_k < r_k < r'_{k+1} < r_{k+1} \\ P_{k-1} < p'_k < P_k < p'_{k+1} < P_{k+1} \end{aligned}$$

where p'_k and p'_{k+1} stand for the probabilities of "greater"-judgments on the new stimuli. It follows from $p'_{k+1} < P_{k+1}$ that there must be also

$$\frac{P_k}{1-P_k} > p'_{k+1}.$$

This means that the position of the maximum is not changed by interpolating new stimuli between the stimulus for which P is a maximum and the subsequent stimulus. This fact is not surprising since no stimulus higher than r_k enters into the formula for P_k . No general statement is possible in regard to the relations

$$\frac{P_{k-1}}{1-P_{k-1}} > P_k \text{ and } p'_k < P_{k-1}$$

and one sees that there may be $\frac{P_{k-1}}{1-P_{k-1}} > p'_k$ as well as

$\frac{P_{k-1}}{1-P_{k-1}} < p'_k$. It must be decided by formula V which of the

values P_k , P'_k and P_{k+1} is the maximum. It follows from these considerations that the most probable result for the determination of the just perceptible difference may be changed by the interpolation of new stimuli, and if it is changed it must become smaller. This indicates that there is a certain danger in taking intermediate steps which are very small and, generally a little vaguely speaking, one ought not to take intervals which are

too small. It will be seen later that the size of the interval depends on the number of experiments one is willing to make and that there exists a principle by which one's choice of the intervals may be guided.

Now let us suppose that the interpolation of new stimuli was carried through in such a way as not to change the position of the maximum of the P 's, but that the number of stimuli preceding this maximum has been increased. This will have the effect of adding some more factors to the product in formula I, and since every factor is smaller than the unit the values of the P 's in the new series must be smaller than those in the first series. This means that it is less likely that an individual result will give the most probable result, which stands out less distinctly in the new series than it did in the first one and the probability of smaller results has increased correspondingly.

These considerations show that the effect of the interpolation of new pairs of comparison stimuli must necessarily be a more or less considerable diminution of the just perceptible difference. By this it is not intended to say that all the observed diminutions of the just perceptible difference are entirely due to this circumstance, because there is some evidence that the attitude of the subject is not quite the same in judging differences in a series with small intervals than it is in series with large intervals. The study of the influence of such a variation of the conditions on attention is a separate problem, but our considerations show that the interpolation of new pairs of comparison stimuli may have the effect of diminishing the just perceptible difference, even if the subject is not influenced at all by the new conditions. The lowering of the threshold is not only not an argument for the ruling out of a particular series, but one would have to be suspicious of a change in the attitude of the subject, if the interpolation of new pairs would constantly fail to have the effect described.

Until now we have confined our discussion to the just perceptible positive difference. The complete method of just perceptible differences requires the determination of three other quantities: The just imperceptible positive difference, the just perceptible negative difference and the just imperceptible negative

difference. All these quantities are expressed by formulae analogous to formula II, so that the theoretical discussions based on this relation are valid for all the quantities in question.

Presenting our series of pairs of stimuli in the order $r_n, r_{n-1}, \dots, r_1$, where $r_n > r_{n-1} > \dots > r_2 > r_1$ we will have a determination of the just imperceptible positive difference in the first stimulus on which the judgment "greater" is not given, all the preceding pairs being judged "greater". Using the same denotation as before we find for the probability that the stimulus r_k will be obtained as a determination of the just imperceptible positive difference

$$P'_k = p_n p_{n-1} \dots p_{k+1} q_k \dots \dots \dots (VI.)$$

because $p_n p_{n-1} \dots p_{k+1}$ is the probability that on all the stimuli greater than r_k a "greater"-judgment will be given and q_k is the probability that such a judgment will not be given on r_k . These probabilities P'_k are different for the different stimuli and in a great number of experiments the most probable result is that every stimulus will be obtained as a determination of the just imperceptible positive difference in a number of times proportional to this probability. The most probable result of these determinations of the just imperceptible positive difference is therefore

$$T' = r_1 P'_1 + r_2 P'_2 + \dots + r_n P'_n \dots \dots \dots (VII.)$$

Formulae VI and VII are analogous to I and II. If the stimulus r_k has the greatest probability for being obtained as a result of a determination of the just imperceptible positive difference we must have

$$P'_{k-1} < P'_k > P'_{k+1}$$

or introducing the expressions for these probabilities

$$p_n p_{n-1} \dots p_k q_{k-1} < p_n p_{n-1} \dots p_{k+1} q_k > p_n p_{n-1} \dots p_{k+2} q_{k+1}$$

which are identical with

$$p_k q_{k-1} < q_k \text{ and } p_{k+1} q_k > q_{k+1} \dots \dots \dots (VIII.)$$

The relations VIII and formula VI may be used for showing that the result of a particular series depends essentially on our

choice of the comparison stimuli, but that it is independent from the order in which the pairs are presented. This demonstration requires only a repetition of our above considerations introducing the p 's in the place of the q 's and vice versa. We will consider only the effect of an interpolation of new pairs of comparison stimuli in that part of a series where the maximum of P'_k is situated. Let us suppose we have three stimuli r_{k-1} , r_k and r_{k+1} for which the probabilities of "greater"-judgments are p_{k-1} , p_k and p_{k+1} , and let their probabilities of being observed as results of the just imperceptible positive difference be P'_{k-1} , P'_k and P'_{k+1} respectively. If P'_k is a maximum we must have the relations

$$q_{k-1} < \frac{q_k}{1-q_k} \quad \text{and} \quad q_k > \frac{q_{k+1}}{1-q_{k+1}}$$

as it follows from the relations VIII. Now let us interpolate a new stimulus r'_{k-1} between r_{k-1} and r_k and another stimulus r'_{k+1} between r_k and r_{k+1} , so that we have

$$r_{k-1} < r'_{k-1} < r_k < r'_{k+1} < r_{k+1}$$

and, therefore, also

$$p_{k-1} < p'_{k-1} < p_k < p'_{k+1} < p_{k+1} \quad \dots\dots\dots (\text{IX.})$$

where p'_{k-1} and p'_{k+1} designate the probabilities of a "greater"-judgment on the comparison of the stimuli r'_{k-1} and r'_{k+1} with the standard. The probabilities of "greater"-judgments and the probabilities of those judgments which are not "greater"-judgments are in the relation $q=1-p$ and it follows from IX that

$$q_{k-1} > q'_{k-1} > q_k > q'_{k+1} > q_{k+1}$$

We may conclude from this relation that

$$q'_{k-1} < \frac{q_k}{1-q_k}$$

but we cannot conclude that

$$q_k > \frac{q'_{k+1}}{1-q'_{k+1}} .$$

From this it follows that the interpolation of new pairs of comparison stimuli has no effect on the most probable value of the

determination of the just imperceptible positive difference, if stimuli are interpolated which are smaller than the stimulus which has the greatest probability. The interpolation of intensities which are larger than the most probable result in the first series may or may not have an effect on the situation of the maximum of probabilities; if the position of the maximum of probabilities has been changed in the new series it must have shifted towards higher values. These considerations bear out the fact, which was observed before, that the result of a determination of the threshold by the method of just perceptible differences depends somewhat on the size of the intervals which are used. It is, therefore, not necessarily a sign of incomplete training of the subject or of his inability to direct his attention on the comparison of the stimuli, if series with small differences fail to give the same result as series with large differences. We supposed in our demonstration that the new series with smaller intervals contained all the stimuli of the old series and some new ones in addition, whereas one frequently will use series of different stimuli. This case was not considered, because it is not possible to make a general statement about the comparative results of two series, unless some other data are at hand. Our method has, furthermore, the advantage of showing the immediate effect of the interpolation of new stimuli.

It depends on the distribution of the P'_k whether the algorithm of the method of just perceptible differences leads to the determination of the most probable value for the just imperceptible positive difference or not. Also in this case it is indispensable to have the results of a great number of different series in order to warrant the hypothesis of a symmetrical distribution. The arithmetical mean has the character of the most probable value if this condition is complied with. The formula for P'_k is exactly analogous to the formula for P_k , the p 's and q 's being interchanged. By repeating our considerations one can show that P'_k is a maximum independent of our choice of the following stimuli if $q_k = \frac{1}{2}$ or since $p_k + q_k = 1$, if $p_k = \frac{1}{2}$. The stimulus for which $p_k = \frac{1}{2}$ is the most probable result for the determination of the just perceptible difference, and we see that the most probable results for the determination of the just perceptible

and of the just imperceptible positive difference are identical. The determination of the just perceptible positive difference is based on the probabilities of the stimuli smaller than r_k and the determination of the just imperceptible positive difference is based on the stimuli beyond this value. One, therefore, must expect to find differences between the results for these values for every particular series of experiments. Taking the average of the just perceptible and of the just imperceptible positive difference for the final determination of the threshold in the direction of increase, has the signification of basing one's determination of the difference for which there exists the probability $\frac{1}{2}$ that a "greater"-judgment will be given on all the results of the series, and not only on the results for the stimuli in the lower part of the series, as the determination of the just perceptible difference does, or on those of the upper part, as the determination of the just imperceptible difference does.

For the determination of the just perceptible and the just imperceptible negative difference we have to consider the probabilities of "smaller"-judgments. Designating by

$$u_1 > u_2 > \dots > u_n$$

the probabilities that in the comparison of the standard with the stimuli $r_1, r_2, \dots, r_n$ a "smaller"-judgment will be given we have

$$\begin{aligned} 1 - u_1 &= v_1 \\ 1 - u_2 &= v_2 \\ &\dots\dots\dots \\ 1 - u_n &= v_n \end{aligned}$$

for the probabilities that a "smaller"-judgment will not be given. The stimulus r_k is a determination of the just perceptible negative difference, if all the stimuli greater than r_k were judged "greater" or "equal" and r_k is judged "smaller". The probability of this compound event is

$$U_k = v_n v_{n-1} \dots v_{k+1} u_k \dots\dots\dots (X.)$$

The most probable result of a great number of determinations of the just perceptible negative difference is

$$S = r_1 U_1 + r_2 U_2 + \dots + r_n U_n \quad \dots\dots\dots (XI.)$$

The just imperceptible negative difference is defined as the smallest intensity on which a judgment is given which is not a "smaller"-judgment, all the stimuli of smaller intensities being judged "smaller." The probability that the stimulus r_k will be obtained as a determination of the just imperceptible negative difference is equal to the compound probability that all the stimuli of smaller intensity are judged "smaller" and that on this stimulus a judgment is given which is not a "smaller"-judgment. This gives

$$U'_k = u_1 u_2 \dots u_{k-1} v_k \quad \dots\dots\dots (XII.)$$

The most probable result of a great number of observations on the just imperceptible negative difference is

$$S' = r_1 U'_1 + r_2 U'_2 + \dots + r_n U'_n \quad \dots\dots\dots (XIII.)$$

Formula X and XII are analogous to those which we obtained for the probabilities of a stimulus being observed as a result of the determination of the just perceptible and of the just imperceptible positive difference. One may find from these formulae the relations which must be satisfied if a stimulus r_k has a maximum of probability. The investigation of these relations leads to the conclusion, that the theoretical values of the just perceptible and of the just imperceptible negative difference are the same. The just perceptible negative difference is that intensity of the comparison stimulus for which there exists the probability $\frac{1}{2}$ that a "smaller"-judgment will be given. The just imperceptible negative difference is that intensity for which there exists the probability $\frac{1}{2}$ that a "smaller"-judgment will not be given. The average of these quantities is a more accurate determination of the intensity for which there exists the probability $\frac{1}{2}$ for a "smaller"-judgment.

The result of the method of just perceptible differences has, therefore, a meaning perfectly well definable in terms of the probabilities of judgments of certain type. We admit three

types of judgments. There is a certain realm for the intensities of the comparison stimuli inside of which neither the judgments "greater" nor the judgments "smaller" have a probability greater than the sum of the two other types of judgments. Outside of this realm, however, one of these judgments has a probability exceeding $\frac{1}{2}$.

We have explained the reasons which suggest the adoption of the method of irregular variation. The results obtained by this method can be worked out by the method of just perceptible differences, if one takes care to record all the judgments given and if all the comparison stimuli are presented to the subject in one round i. e. that a pair is not presented a second time before all the other pairs were presented. From the records one picks out the stimulus which is the smallest on which a "greater"-judgment was given and the largest on which such a judgment was not given. The first serves for the determination of the just perceptible positive difference, the second for the determination of the just imperceptible positive difference. The next step is to find the greatest stimulus on which a "smaller"-judgment is given and the smallest stimulus on which such a judgment is not given; these stimuli are determinations of the just perceptible and of the just imperceptible negative difference. The complete records, therefore, may be worked out by the method of just perceptible differences and, since they give the percentages of correct and wrong judgments, also by any one of the error methods. The necessity of recording all the judgments given is avoided in that form of the method of just perceptible differences, which is recommended by most writers, where the outcome of an entire series is reduced to one number. This little saving of clerical work is quite inconsiderable when compared with the work actually spent in experimenting, and it is by far outweighed by the possibility of impairing the results through the influence of expectation on the part of the subject. In the complete records one also has data which may serve other purposes than the determination of the threshold. The possibility of presenting the pairs in different order in every round, furthermore, gives a means of eliminating influences due to the arrangement of the series.

Under the supposition that the judgment on one pair is not influenced by the judgments on any of the preceding pairs, i. e. that the order in which the pairs are presented is irrelevant, the result of the method of just perceptible differences is identical with that of the method of irregular variation. Both methods determine the intensities for which there exists the probability $\frac{1}{2}$ that on the comparison with the standard stimulus the judgments "greater" or "smaller" will be given. These methods are entirely independent of any hypothesis as to the law of distribution of correct and wrong cases, and this is a great superiority over the method of right and wrong cases. The essential feature of the error methods is that one attempts to find certain constants, which enable one to find the relative frequency of correct cases for any difference from the observed relative frequency of these judgments on one difference or on several differences, as it was suggested by Müller and Titchener. It makes but little difference whether one uses for this purpose the simple law of Gauss, or Fechner's double sided law of Gauss, or whether one may use one of Pearson's formulae or any other relation, which by its nature is fit to represent a law of distribution. Among all these different distributions the law of Gauss must necessarily retain a very prominent place for all practical purposes, not only because it depends only on one parameter, but also because it is until now the best understood of all the laws of distribution and finally, because it does not suffer from little shortcomings as e. g. the discontinuity of the second derivative in Fechner's generalisation of this law. The method of right and wrong cases may be characterized as a method of interpolation under the supposition of a particular law of distribution. The scope of this method is by far more ambitious than that of the method of just perceptible differences or of the method of irregular variation, but it can be reached only by means of a definite hypothesis which may be dispensed with for either one of the two methods.

It still remains to consider what the limits are inside of which we may expect the result of the method of just perceptible differences with a given probability. It seems very obvious to formulate the problem in this way:

The result r_1 may be expected with the probability P_1 ;
 " " r_2 " " " " " " P_2 ;

 The result r_n may be expected with the probability P_n .

One makes N trials each one of which must result in r_1 , or in r_2 , or or in r_n . Which is the probability that the sum of all the results will be S ? One may try to find the most probable result and the limits within which one may expect a deviation with a given probability, just in the same way as one derives these quantities for Bernoulli's theorem. This formulation is nothing but a generalisation of the so-called problem of Moivre. The algebraical difficulties of this problem soon become very great and it does not seem that there exists at present a general solution. In any case it is not likely that the solution will be simple.

The second way of formulating the problem starts from the consideration that the most probable outcome of a series of N experiments will be that r_1 occurred NP_1 times, r_2 occurred NP_2 times and so on, and that the theorem of Bernoulli gives the probabilities for the deviations from these most probable results. One therefore may compute the probable deviations and find the probable error of the final result from them. This way leads to a result, but it seems to be the most elegant and most profitable way to start from our first interpretation of formula II and to apply the following theorem of Tchebitcheff.* Let $x, y, z, \dots$ be any magnitudes which may assume different values, each one with a certain probability, and let the values of x be

$$x_1, x_2, \dots, x_k$$

to which correspond the probabilities

$$p_1, p_2, \dots, p_k$$

so that

$$\sum_{i=1}^k p_k = 1;$$

*TCHEBITCHEFF, *Journal de Liouville*, (2) Vol. 12, 1867, p. 177.

in the same way let the values of y be

$$y_1, y_2, \dots, y_l$$

with the probabilities

$$q_1, q_2, \dots, q_l$$

so that

$$\sum_1^l q_k = 1,$$

and so on. We then have

$$\sum_1^k p_i x_i = a, \quad \sum_1^l q_k y_k = b, \quad \dots\dots$$

for the mathematical expectations or averages of $x, y, \dots$, and

$$\sum_1^k p_i x_i^2 = a_1, \quad \sum_1^l q_k y_k^2 = b_1, \quad \dots\dots$$

for the mean values of the squares. Under these conditions the theorem holds, that the probability P that the arithmetical mean

$$\frac{x_k + y_k + \dots}{n}$$

of the observed values of $x, y, z, \dots$ is contained between the limits

$$\frac{a+b+\dots}{n} - \frac{1}{t} \sqrt{\frac{a_1+b_1+\dots}{n} - \frac{a^2+b^2+\dots}{n}}$$

and

$$\frac{a+b+\dots}{n} + \frac{1}{t} \sqrt{\frac{a_1+b_1+\dots}{n} - \frac{a^2+b^2+\dots}{n}}$$

is larger than $1 - \frac{t^2}{n}$.

This theorem may serve for the determination of the accuracy of the results of the method of just perceptible differ-

ences. Let us suppose that there were h different series of comparison stimuli, the first arrangement consisting of the comparison stimuli

$$r_1, r_2, \dots, r_k$$

which have the probabilities for being obtained as determinations of the just perceptible difference

$$P_1, P_2, \dots, P_k;$$

in the second arrangement the stimuli

$$r'_1, r'_2, \dots, r'_1$$

were used which had the probabilities

$$P'_1, P'_2, \dots, P'_1$$

respectively, for being obtained as determinations of the just perceptible difference; and so on. We then have

$$\sum_1^k r_i P_i = a_1, \sum_1^1 r'_k P'_k = b, \dots$$

for the mathematical expectations in the single series, and

$$\sum_1^k P_i r_i^2 = a_1, \sum_1^1 r'_k P'_k = b, \dots$$

for the mean values of the square. With the first arrangement we make n experiments, with the second n' experiments, with the third n'' experiments, ..., so that

$$n + n' + n'' + \dots = N.$$

It will have the same effect for the calculation if we take the mean value and the mean square for each series as often as this series was given. We then have a probability P that the arithmetical mean

$$\frac{n r_k + n' r'_k + \dots}{N}$$

of the observed values will be contained between the limits

$$\frac{na+n'b+\dots}{N} + \frac{1}{t} \sqrt{\frac{na_1+n'b_1+\dots}{N} - \frac{na^2+n'b^2+\dots}{N}}$$

and

$$\frac{na+n'b+\dots}{N} - \frac{1}{t} \sqrt{\frac{na_1+n'b_1+\dots}{N} - \frac{na^2+n'b^2+\dots}{N}}$$

which is greater than $1 - \frac{t^2}{N}$. This formula shows the advantage of using great numbers of experiments, because the difference $1 - \frac{t^2}{N}$ approaches the unit if N increases and t remains constant. This means that it is possible to choose the number of experiments so large that one may expect with a probability as little different from the unit as one pleases, that the actual result will be within given limits from the calculated result.

In regard to the limits of accuracy it may be allowed to make the following comparison of the method of just perceptible differences with the error methods. The limits of accuracy of an empirical determination of a probability depend on the coefficient of precision in Bernoulli's theorem, which is a function of the probability of the event. The coefficient of precision has a minimum for the probability $\frac{1}{2}$, so that events which may be compared with the tossing up of a coin have the smallest precision. The coefficient of precision increases as the value of the probability of the event approaches zero or the unit. The accuracy of the method of just perceptible differences depends on empirical determinations of the P 's, which are constituted of the p 's which are used in the error methods. Formula II shows that the P 's are always smaller than the corresponding p 's for the same stimulus except the trivial case $k=1$ where they are equal. This shows that in the observation of events based on the P 's a greater precision may be expected than for a series of observations based on the p 's. The method of just perceptible differences and the method of irregular variations make use of the P 's, but the error methods use the p 's, so that a number of observations made by the method of right and wrong cases will

give a smaller precision than the same number of observations by the method of just perceptible differences or by the method of irregular variation. These methods do not undertake as much as the method of right and wrong cases does; but what they do they do with a great amount of precision.

On the basis of these considerations we may give the following description of the method of just perceptible differences. Prepare a series of comparison stimuli which cover an interval in the beginning of which the subject gives a very high percentage of "smaller"-judgments and at the end of which there exists a similar high probability for "greater"-judgments. Be careful to apply the standard stimulus and the comparison stimulus always in the same order so as to keep the time error constant. Present the pairs in any order and record all the judgments given. If the investigation is somewhat extended it is necessary to exhaust the possible orders systematically, so as not to favor any single one. Use different arrangements of comparison stimuli. All the stimuli of the same arrangement must be gone through, before the same stimulus is presented to the subject another time. Find from the records the smallest comparison stimulus in every series on which the judgment "greater" was given, and the largest stimulus on which another but a "greater"-judgment was given. A comparison stimulus is the smallest in a series to be judged "greater", if all the comparison stimuli of smaller intensity were judged "smaller" or "equal" and if this stimulus was judged "greater." A comparison stimulus is the largest on which a "greater"-judgment was not given, if on this stimulus either one of the judgments "smaller" or "equal" is passed and if all the larger comparison stimuli were judged "greater". These are the data for the determination of the just perceptible and of the just imperceptible positive difference. Combine the results of the different series by taking the arithmetical mean of all the observations for the determination of the just perceptible positive difference and for that of the just imperceptible positive difference. The average of the just perceptible and of the just imperceptible positive difference gives that amount of difference, for which there exists the probability $\frac{1}{2}$ that the judgment "greater" will be

given. This value is called the threshold in the direction of increase. Find from the same records the largest stimulus on which the judgment "smaller" was given, and the smallest stimulus on which a "smaller"-judgment was not given. A stimulus is the largest of a series on which the judgment "smaller" was given, if this stimulus was judged smaller and if all the larger differences were, judged "equal" or "greater". A stimulus is the smallest on which the judgment "smaller" was not given, if this stimulus was judged "equal" or "greater" and if all the smaller stimuli were judged "smaller". Find the just perceptible and the just imperceptible negative difference by taking the averages of all the observations and combine the results. The final result is that amount of difference for which there exists the probability $\frac{1}{2}$ that the judgment "smaller" will be given. This quantity is called the threshold in the direction of decrease.

The result of the so-called method of just perceptible differences has, therefore, a signification which does not depend on the notion of a difference which is always perceived. The result of this method is expressible in the fundamental terms of the error methods and it was seen that this method has several decisive advantages over the other methods which use the same type of experiments for the measurement of sensitivity. The method of just perceptible differences has the feature of an eminently practical method, in so far as the experiments required are very simple, as its algorithm is extremely easy and its degree of accuracy is great, so that only a relatively small number of observations is required. For all the ordinary purposes of a determination of the threshold one may use the method in the form described by the previous investigators and one will obtain serviceable results. For all purposes, however, which require some accuracy one will not fail to keep a complete record of all the judgments given. It will be seen later that such a set of results can be treated more exhaustively by another method, which combines the advantages of the error methods with those of the method of just perceptible differences and which yields the result of the latter method almost without any work.

This description of the method of just perceptible differences

may be illustrated by the example of our experiments on lifted weights. The judgments "greater" and "smaller" correspond to our judgments "heavier" and "lighter", and the "guess"-judgments of our experiments are equivalent to the equality cases. From the observed frequencies of these three types of judgments we may find their probabilities for the differences which were used. The theorem of Bernoulli gives the limits of accuracy of these determinations. These probabilities are the p 's in our formulae and we may find from them the value of the P 's for every difference used in the experiments by formula II. The first business is to arrange the results in the way shown in Table 28 (Appendix p. 186). Each one of our seven subjects is given a double column, the first column of which bears the heading "h" and the second the heading "not-h." The column "h" gives the observed relative frequencies of "heavier"-judgments for every difference, and the second column gives the difference of this number from the unit, i. e. the relative frequencies of judgments which are not "heavier"-judgments. From these numbers one finds the values of the P 's, beginning at the top of the table, successively; these values are given in Table 29 (Appendix p. 186). This computation does not take much time if one considers that the product in formula II contains the q 's for all the preceding stimuli. One divides conveniently the computation in two steps, the first of which consists in finding the sum of the logarithms of the q 's and the second in adding the logarithm of the corresponding p_k . A glance at the results in Table 29 shows that the numbers increase at first and decrease after having attained a maximum. The rise and fall of the numbers is uniform not being interrupted by secondary maxima except for subject V where there is a slight notch at 88. The absence of interruptions is doubtlessly due to the fact that the intervals between the comparison stimuli used in the experiments are rather large. It will be noticed furthermore, that the maximum of the P 's coincides with one of the stimuli at the beginning or at the end of the interval in which the value $p = \frac{1}{2}$ is found. It so happens that the maximum is at the end of the interval for the subjects I to VI, and that the maximum is found at the beginning of the interval only for the subject

VII. The sum of all the P 's is given at the bottom of the table. These sums are slightly different from the unit, which indicate that there is a probability which is not negligible that the entire series may be presented to the subject without giving a result for the determination of the threshold. This circumstance affords a very convenient way of checking the result of the computation. A series of experiments will not give a result for the determination of the just perceptible positive difference, if no "heavier"-judgment is given on any one of the stimuli. The probability of this event is identical with the probability that on all the stimuli judgments are given which are not "heavier"-judgments, and this probability is equal to the product of all the q 's. For the computation of the last P one has to form the product of all the q 's with the exception of the q for the last stimulus. The product of all the q 's, therefore, may be found by one single multiplication and the result must add up to one with the sum of all the P 's. The products of the q 's are given in the line marked R . The sum of Σ and R is equal to the unit except in those cases where it was necessary to correct the individual values of the P 's in such a way that their sum involved an error of a unit at the last decimal place. We may make the remark that it is necessary to start out with four correct decimals in order to get the final results correct within two decimals.

The next step is to find the values of $r_k P_k$ and of $r_k \cdot P_k$, which also may be done conveniently by successive multiplication. Table 30 (Appendix p. 187) gives the value of $r_k P_k$. The numbers of this table show that the significant values are clustered around a certain point, which is different for the different subjects, and that the values of these products decrease very rapidly on both sides of this point. This indicates that the influence of adding new stimuli at the ends of the series is inconsiderable. It seems, however, that our series is not extended enough for a determination of the just perceptible positive difference, because the values at the bottom of the table are considerably greater than those at the upper end of the table. We conclude that it would be advisable in a series of experiments which has the main purpose of ascertaining the threshold of difference to add at

least one more stimulus at the upper end of our series of stimuli. This is also suggested by the fact that the values of R in Table 29 are by no means inconsiderable, so that the chances are that one will not only have a relatively great number of resultless experiments, but one also will neglect some terms which are of influence for the determination of the threshold of difference. The value of $r_k P_k$ for $r=108$ gives an indication of the possible influence of the next term and the following comparison shows that the values of R and influence of the last term go parallel to some extent.

Subject	Value of R	$r P$ for 108 gr.
VII	0.0189	9.2772
VI	0.0128	10.4976
II	0.0050	5.0868
I	0.0025	4.3092
III	0.0017	4.1148
IV	0.0017	2.4732
V	0.0009	1.6524

It is advisable to have in every series at least two stimuli for which the probability of a "heavier"-judgment is very close to the unit because the product of the q 's becomes very small in this case. A further observation which is impressed by Table 30 is the marked asymmetry in the distribution of the values. With the exception of subject V the values of $r_k P_k$ increase slowly and show a very steep descent, whereas in the subject V the course is opposite. The skewness is quite unmistakable also in this case, but the slope in the ascent and descent of the values is not as steep for this subject as the descent of the values for the other subjects.

The sum of all the $r_k P_k$'s is given at the bottom of Table 30. This value is the just perceptible positive difference for this series. If we were dealing with symmetrical distributions this value would give the amount of difference for which there exists the probability $\frac{1}{2}$ that it will be judged "greater", but independently from a supposition about the form of the distribution this number has the signification of the mathematical expectation. It is noteworthy that all these values are smaller than 100 gr, which

indicates that in the comparison of a 100 gr standard with a 100 gr comparison weight a number of "heavier"-judgments will be given which exceeds 50%. The numbers in our table vary between 96.9139 and 99.8164, and it is, perhaps, not a chance incident that the smallest value is that of the female observer. Her value is by more than 1 gr smaller than the lowest value of the other subjects. From the values of $r_k P_k$ given in Table 30 we find the values of $r_k^2 P_k$ immediately. These numbers and their sums are given in Table 31 (Appendix p. 187). The sum of these numbers gives the mean value of the squares, a number which is needed for the application of the theorem of Tchebitcheff. Subtracting from the sums in this table the square of the numbers at the bottom of the preceding table and taking the square root from the result gives the coefficient which we need for the computation of the probability of a deviation by the theorem of Tchebitcheff. We obtain in this way the numbers 15.11, 8.64, 5.97, 6.23, 5.20, 12.30 and 14.41 for our seven subjects I, II, ..., VII. The most probable result for the determination of the just perceptible positive difference and the probabilities for deviations from it may be represented in this way: There exists a probability greater than $1 - \frac{t^2}{n}$ that the observed value will fall within the limits

$$99.76 - \frac{15.11}{t} \quad \text{and} \quad 99.76 + \frac{15.11}{t} \quad \text{for the subject} \quad \text{I};$$

$$98.31 - \frac{8.64}{t} \quad " \quad 98.31 + \frac{8.64}{t} \quad " \quad " \quad " \quad \text{II};$$

$$99.82 - \frac{5.97}{t} \quad " \quad 99.82 + \frac{5.97}{t} \quad " \quad " \quad " \quad \text{III};$$

$$97.98 - \frac{6.23}{t} \quad " \quad 97.98 + \frac{6.23}{t} \quad " \quad " \quad " \quad \text{IV};$$

$$96.91 - \frac{5.20}{t} \text{ and } 96.91 + \frac{5.20}{t} \text{ for the subject V;}$$

$$99.21 - \frac{12.30}{t} \quad " \quad 99.21 + \frac{12.30}{t} \quad " \quad " \quad " \quad \text{VI;}$$

$$98.76 - \frac{14.41}{t} \quad " \quad 98.76 + \frac{14.41}{t} \quad " \quad " \quad " \quad \text{VII.}$$

These numbers may be used in two ways; either one may ask how many experiments are needed in order to find the result inside of a certain interval with a given probability, or one may ask what is the probability that a given deviation will be observed in a certain number of experiments. This affords the possibility of comparing the theoretical results with the actual observations. We may ask, for instance, what is the interval inside of which we may expect the result with the probability $\frac{1}{2}$, if for each of the subjects I, II and III 400 and for every other subject 300 observations on the just perceptible positive difference were made. For the first three subjects we find $t = \sqrt{200} = 14.14$ and for the other subjects $t = \sqrt{175} = 13.23$. The observed differences from the theoretical value may be found from Table 39 (Appendix p. 191), and they may be compared with the results of the computation. The deviation from the most probable result inside of which the observed result may be expected with a probability greater than $\frac{1}{2}$ is

for the	I	subject	1.07,	the observed	difference is	0.60
" "	II	"	0.61	"	"	0.40
" "	III	"	0.42	"	"	0.32
" "	IV	"	0.47	"	"	0.54
" "	V	"	0.55	"	"	0.78
" "	V	"	0.55	"	"	0.78
" "	VI	"	0.91	"	"	1.20
" "	VII	"	1.09	"	"	2.11.

Four of the observed differences are greater and three are smaller than the theoretical result. We would have to expect such a result for an event the probability of which is approximately $\frac{1}{4}$. It is noteworthy that the observed and the theoretical deviations are of the same order of magnitude.

In order to obtain the empirical determination of the just perceptible positive difference we have to go back to the records of the experiments, and we have to find in every column that weight which was the smallest on which a "heavier"-judgment was given. In the example of a record sheet which we have given on page 6 we find in the first column 92, in the second column 96, in the third column 96, in the fourth column 100 and in the fifth column 96 as the smallest weight on which the judgment "heavier" is given. In this way one may find for each comparison weight, how many times it happened that it was judged "heavier" when all the smaller weights were judged "lighter" or "equal". These results are given in the Tables 32-38 (Appendix pp. 188-191). Each subject is given a separate table, which contains the results of every series of 100 experiments in a double column. We have, therefore, 400 determinations of the just perceptible positive difference for the subjects I, II, III and 300 determinations for each of the other subjects. The numbers in the columns under the heading N_k give for every weight the number of times it was obtained as a determination of the just perceptible positive difference, and the numbers opposite in the next column give the product of the weight multiplied by the number of times it was observed as the just perceptible difference ($r_k N_k$). This is the number which must be taken for every weight according to the algorithm of the method of just perceptible differences. The final determination is obtained by dividing the sum of the $r_k N_k$ by the total number of observations, i. e. by the sum of the N_k . The sums are given at the bottom of the tables in the line marked Σ and the results of the final determination of the just perceptible positive difference are given in the line marked "Average". It will be remarked that the sum of the N_k is not always equal to 100. This is a consequence of the fact that the sum of the P_k is not exactly equal to the unit. The number of those cases which did not give a result for the determination of the just per-

ceptible positive difference is small, as it may be expected since the smallness of the differences of the sums of the P's from the unit indicates, that it is a rather rare event that all the pairs are presented to the subject without a single weight being judged "heavier". This event happened most frequently with the subjects II, VI and VII, for whom the difference of the sum of the P's from the unit is largest. The comparison of the observed results for the determination of the just perceptible difference with the calculated results given in Table 30 shows that the coincidence of the observed results with the theoretical results is very satisfactory even with a relatively small number of observations. An analysis of this coincidence is given in Table 39 (Appendix p. 191), where it is shown how much the result of the observations differs from the theoretical result in every series of 100 experiments. The deviations are very small and they comply with the test of Tchebitcheff's theorem in a very high degree. In a small majority of cases the observed result is found inside the interval for which there exists a probability greater than $\frac{1}{2}$. The last column of this table gives the combined result of all the observations. The numbers are found from the data of Table 40 (Appendix p. 192), which shows the results of the determination of the just perceptible difference if the results of all the series are combined. The signification of N_k and $r_k N_k$ is the same as in the previous tables. Every subject is given a double column and the numbers N_k in this table are equal to the sum of the corresponding values given above in the table for the same subject. A test of the correctness of the computation consists in comparing the sum of $r_k N_k$ which is equal to the sum of the corresponding values in the former table for the same subject. In close agreement with the results which were found above we find in the majority of cases the value of the just perceptible positive difference below 100, and in those cases where it exceeds 100 the difference is very small. The term "just perceptible positive difference" is, therefore, somewhat misleading, because the actual difference between the two stimuli will be negative in most cases. This is due to the fact that the time error was not eliminated in our experiments. One must keep in mind that we use the term "just perceptible positive difference" merely as an abbreviation

for the comparison stimulus of the pair, which is obtained as a result of a determination of the just perceptible positive difference by the method of just perceptible differences. We know that there exists for this stimulus the probability $\frac{1}{2}$ that the comparison with the standard will result in the judgment "greater". It is not to be feared that a serious misunderstanding may arise from this terminology, and for this reason it does not seem advisable to change a terminology which is more or less commonly in use. The negative sign of this difference indicates that one must not start from objective equality of the stimuli when applying the method of just perceptible differences but from subjective equality, a fact which was recognised also by previous investigators of this method. The judgment which is passed on the comparison of the stimuli depends on a number of conditions among which the actual difference of the stimuli is only one. We have to distinguish between the actual difference and the effective difference, to use a term of G. E. Müller, understanding by this term the difference of a stimulus from subjective equality.

The next step to be taken in working out the data by the method of just perceptible differences is the computation of the just imperceptible positive difference. The necessary data for this computation are given in Table 28. Beginning at the bottom of the table we have to form successively the products of the p 's and multiply them by the corresponding values of q . In this way one finds the numbers given in Table 41 (Appendix p. 192), which are the P 's for the computation of the just imperceptible positive difference. The P 's for 108 are, of course, identical with the probability that on this weight a judgment will be given which is not a "heavier"-judgment. The course of the numbers in this table is similar to that of the numbers in Table 29. It is perfectly regular, not a single break in the ascent or descent occurring throughout the entire table. In some cases the ascent is very rapid the maximum being reached almost with a jump (subject I, III and IV), in other cases the ascent and descent take place with approximately equal rapidity (e. g. subject V), so that without further investigation one almost might speak of a symmetrical distribution. The values of the P 's become very small for 88 and 84, and since the values for 108 are rather large,

it makes the impression as if the table of the distribution of the P's were cut off at the upper end. A consequence of the small values of P for the last differences of the table is that the values of R are insignificant for all the seven subjects. In no case has R a counting figure on the fourth decimal place. The sum of the P's may differ by the unit of the last decimal place from one owing to necessary corrections of the single P's.

Multiplying the numbers of Table 41 with the corresponding r's one finds the values $r_k P_k$, which are given in Table 42 (Appendix p. 193). It becomes evident from this table that the series of comparison stimuli is extended enough in the direction of decrease, but that it is too short in the upper part. The lower part of this table shows very well how rapidly the influence of a comparison stimulus becomes insignificant, if it is somewhat distant from the most probable result. The sum of all the terms $r_k P_k$ gives the just imperceptible positive difference, which is given at the bottom of Table 42 in the line marked Σ . It will not be void of interest to compare the theoretical values of the just perceptible and of the just imperceptible positive differences.

Subject.	Just perceptible positive difference.	Just imperceptible positive difference.
I	99.7642	99.1388
II	98.3050	99.3640
III	99.8164	98.7522
IV	97.9791	98.1866
V	96.9139	97.3702
VI	99.2082	100.5176
VII	98.7590	100.9592

These numbers show that the just imperceptible positive difference is smaller than the just perceptible positive difference in the case of two subjects (I and III) and that it is larger in all the other cases. Several investigators have made the observation that, with their subjects, the just imperceptible positive differences had the tendency to be smaller than the just perceptible positive differences. This fact was attributed to the influence of attention and it seemed to be an objection against the method of just perceptible differences. Our results bear out, what

one easily can verify by theoretical considerations, that in some cases the just perceptible positive difference will be greater than the just imperceptible difference and that in other cases the opposite will take place. It depends on the chance influence of the choice of the comparison stimuli which difference will be greater.

Multiplying the values $r_k P_k$ by r_k gives the values $r_k^2 P_k$ which are given in Table 43 (Appendix p. 193). The mean value of the squares is needed for the determination of the limits inside of which the result may be expected to fall with a probability exceeding $1 - \frac{t^2}{n}$. These intervals are

$$99.14 - \frac{3.86}{t} \text{ and } 99.14 + \frac{3.86}{t} \text{ for the subject I;}$$

$$99.36 - \frac{4.58}{t} \text{ " } 99.36 + \frac{4.58}{t} \text{ " " " II;}$$

$$98.75 - \frac{3.71}{t} \text{ " } 98.75 + \frac{3.71}{t} \text{ " " " III;}$$

$$98.19 - \frac{4.64}{t} \text{ " } 98.19 + \frac{4.64}{t} \text{ " " " IV;}$$

$$97.37 - \frac{5.10}{t} \text{ " } 97.51 + \frac{5.10}{t} \text{ " " " V;}$$

$$100.52 - \frac{4.24}{t} \text{ " } 100.52 + \frac{4.24}{t} \text{ " " " VI;}$$

$$100.96 - \frac{4.52}{t} \text{ " } 100.96 + \frac{4.52}{t} \text{ " " " VII.}$$

The coefficients which determine the size of the interval are smaller for the just imperceptible positive difference than for the just perceptible positive difference, and we may expect that

the coincidence of the observed results with the theoretical results is even closer than it was in the first case.

For the empirical determination of the just imperceptible positive difference we have to find from the records how many times every weight was the largest on which a judgment was given which was not a "heavier"-judgment. In the example of a record sheet on p. 6 we have 100 in the first column, 92 in the second column, 92 in the third column, 96 in the fourth column and 92 in the last column as empirical determinations of the just imperceptible positive difference. Counting the records over in this way one obtains the data for the construction of the Tables 44-50 (Appendix pp. 194-197), which are similar in their construction to the Tables 32-38. Each subject is given a separate table, which contains in every double column the results of a series of 100 experiments. The numbers in the column N_k refer to the number of times a weight was obtained as a determination of the just imperceptible positive difference. The numbers in the columns $r_k N_k$ give the product of the weight with the number of times it was obtained as a determination of the just imperceptible positive difference. The numbers at the bottom of the tables give the sums of all the numbers in the corresponding columns. The averages of the numbers $r_k N_k$ are the empirical determinations of the just imperceptible positive difference in these particular series. The results of the observations agree very well not only with the theoretical results but also with the results of the empirical determinations of the just perceptible positive difference. We have seen that in the case of a symmetrical distribution the just perceptible positive difference is that amount of difference for which there exists the probability $\frac{1}{2}$ that a "heavier"-judgment will be given, and that under the same condition of a symmetrical distribution the just imperceptible positive difference is that amount of difference for which there exists the probability $\frac{1}{2}$ that a "heavier"-judgment will not be given. This is of course the same quantity in both cases and empirical determinations of the just perceptible positive difference and of the just imperceptible positive difference must give results which are sensibly equal. The comparison of these numbers is given in Table 51 (Appendix p. 197), which contains in every

double column the results for the just perceptible and for the just imperceptible positive difference of one series of 100 experiments. The columns under the heading A give the results of the determination of the just perceptible positive difference, and those under the heading B give the just imperceptible positive difference. The largest difference between the corresponding numbers in the columns A and B is 2.92 gr (subject III, series I), and the smallest is 0.04 gr for the same subject in the series IVa and in the series IV. The differences between these values are not large, and small differences, furthermore, interchange with large differences; it is, however, remarkable that the just perceptible difference is greater than the just imperceptible difference in a large majority of the cases.

The results of all the series of 100 experiments each may be combined and a determination of the just imperceptible difference may be obtained from all the data. The combined results are given and worked out in Table 52 (Appendix p. 198), which is constructed in the same way as Table 40 giving the results for each one of the 7 subjects in one double column. It is of interest to notice that the sum of the N's differs from the total number of experiments for two subjects (IV and VII). This indicates that it is rather rare that the series is gone through without one weight being judged "not-h." This agrees with Table 41 although the observed values are somewhat larger than we may expect on the basis of the numbers of this table. The results of the final determination of the just imperceptible positive difference compare in this way with those for the just perceptible positive difference.

Subject.	Just perceptible positive difference.	Just imperceptible positive difference.	Difference.
I	100.36	98.84	+1.52
II	98.71	98.71	0.00
III	99.50	99.67	-0.17
IV	98.52	97.97	+0.55
V	97.69	97.00	+0.69
VI	100.41	100.25	+0.16
VII	100.87	98.37	+2.50

A striking feature of this comparison is the sign of the differences. The just perceptible positive difference is almost invariably larger than the just imperceptible positive difference. This observation agrees with the results of previous investigators of the method of just perceptible differences, who as a rule attributed this fact to the particular order in which the comparison stimuli were presented. It was supposed that a small stimulus was more readily perceived, if it was preceded by several other small stimuli of decreasing intensity, because attention was better adapted by the preceding stimulation. This explanation can not hold for our experiments because the stimuli were presented in very different order and an adaptation of attention, therefore, cannot take place.

We now turn to the description of our observations on the just perceptible and on the just imperceptible negative difference. The empirical determinations of the probabilities of a "lighter"-judgment are the data which we need for this computation. These numbers are given in Table 53 (Appendix p. 198) for each subject in one double column. The numbers in the columns marked "1" give the probabilities of "lighter"-judgments for the various differences. The numbers in the columns marked "not-1" give the probabilities of judgments which are not "lighter"-judgments. These probabilities are of course equal to the differences of the numbers "1" from the unit. The numbers "1" and "not-1" are the values u and v in our theoretical deductions and we find from them the U 's by successive multiplication beginning at the bottom of the table. The values of U and u are identical for the comparison weight 108 gr, as may be seen from Table 54 (Appendix p. 199), where the values of the U 's are given. The increase and decrease of the U 's is uniform throughout the whole table. The sums of the U 's are very close to the unit as may be seen from the values of R , which are given at the bottom of table. It is instructive to compare the values of R in this table with those of Table 29.

Subject.	Value of R in Table 29.	Value of R in Table 54.
I	0.0025	0.0022
II	0.0050	0.0010
III	0.0017	0.0000
IV	0.0017	0.0003
V	0.0009	0.0005
VI	0.0128	0.0004
VII	0.0189	0.0002 .

The values of R for the just perceptible positive difference are throughout greater than those for the just perceptible negative difference, and only for the first subject does the latter value come anywhere near the first. This indicates that the series of comparison weights is better adapted for the determination of the just perceptible negative difference. The largest value of R in Table 54 is 0.0022 or approximately $\frac{1}{500}$. Since R gives the probability that the series of stimuli will be presented to the subject without giving a result for the method of just perceptible differences — a fact which one will try to avoid for obvious reasons — one may be satisfied with this series of stimuli in a determination of the threshold which does not aim at a higher accuracy than it is attainable in some 500 experiments. For the subjects III and IV one even could leave out the comparison stimulus 84 without seriously impairing the efficiency of the series. This observation indicates that the series of comparison stimuli must be adapted to the individual whose sensitivity is to be tested, and that superfluous experimenting may be avoided by an appropriate choice of the stimuli. The values of the u 's serve for the computation of the products $r_k U_k$ and $r_k^2 U_k^2$, which are given in Tables 55 and 56 (Appendix p. 199 sq). which are needed for the calculation of the just perceptible negative difference and for the determination of the reliability of this determination. The data of these two tables allow to represent the determination of the just perceptible negative difference in the following way. There exists a probability greater than $1 - \frac{t^2}{n}$ that the result of an observation of this quantity will fall within the limits

$$93.08 - \frac{6.30}{t} \text{ and } 93.08 + \frac{6.30}{t} \text{ for the subject I;}$$

$$95.28 - \frac{7.89}{t} \quad " \quad 95.28 + \frac{7.89}{t} \quad " \quad " \quad " \quad \text{II;}$$

$$97.40 - \frac{3.89}{t} \quad " \quad 97.40 + \frac{3.89}{t} \quad " \quad " \quad " \quad \text{III;}$$

$$95.23 - \frac{5.74}{t} \quad " \quad 95.23 + \frac{5.74}{t} \quad " \quad " \quad " \quad \text{IV;}$$

$$94.20 - \frac{5.21}{t} \quad " \quad 94.20 + \frac{5.21}{t} \quad " \quad " \quad " \quad \text{V;}$$

$$95.25 - \frac{5.12}{t} \quad " \quad 95.52 + \frac{5.12}{t} \quad " \quad " \quad " \quad \text{VI;}$$

$$96.04 - \frac{5.08}{t} \quad " \quad 94.04 + \frac{5.08}{t} \quad " \quad " \quad " \quad \text{VII.}$$

These values are found in the same way as the corresponding values previously given, namely by the algorithm

$$\Sigma r_k P_k \pm \frac{1}{t} \sqrt{\Sigma r_k^2 P_k - (\Sigma r_k P_k)^2}.$$

The signification of these numbers is similar to that of the numbers given in Tables 30 and 31, so that it does not need be discussed any more. We proceed immediately to give the results of the observations on the just perceptible negative difference. A stimulus will be recorded as an observation of the just perceptible negative difference, if it is the largest on which the judgment "lighter" was given, all the larger stimuli being judged "heavier" or "equal." In the sample of a record sheet given

on p. 6 we have in the first column 88, in the second 84, in the third column 92, in the fourth column 92 and in the fifth columns 88 as determinations of the just perceptible negative difference. These results of our experiments are given in the Tables 57-63 (Appendix pp. 200-203), which are constructed in the same way as the corresponding tables for the just perceptible positive difference. The results for each subject are given in a separate table, which contains the results of every series of 100 experiments in one double column. The headings N_k and $r_k N_k$ have the same signification as before. The sums of N_k and those of the products $r_k N_k$ are given at the bottom of every column and the final result of the determination of the just perceptible negative difference is given in the line marked "Average". The following table (Table 64, Appendix p. 204) gives the determination of the just perceptible negative difference if all the results are combined. Our theoretical considerations have shown that we may expect the results to fall within rather narrow limits. The results verify this expectation in so far as our observations show not only a close coincidence with the theoretical results, but the results of the different series of 100 experiments of the same subject also show very little variation from each other. This coincidence is illustrated by the data of Table 65 (Appendix p. 205), which gives under the heading IVa, I, III, IV the results for the different series, and in the column "Total" the result of the determination of the just perceptible negative difference for the combined results. If the coincidence of the observed results with the theoretical results is great in the series of 100 experiments one cannot help calling this coincidence for the more extended series surprising. The maximum of the deviations is 0.18 gr, the minimum is 0.00 and the average is 0.076 gr. The number of the positive and that of the negative deviations are approximately equal. This shows that the method of just perceptible differences is capable of great exactitude under favorable circumstances.

We now turn to the study of the just imperceptible negative difference, the data for the computation of which we find in Table 53 (Appendix p. 198). Applying the formula based on the definition of the just imperceptible negative difference one finds

by successive multiplication, beginning at the top of Table 53, the values of the U's, which are given in Table 66 (Appendix p. 205). The values of U for 84 are identical with the probabilities of a "not-1"-judgment for this weight. The sums of the U's are only slightly different from the unit, indicating that the probability is very small that a series will end resultless. In one case (subject I) it happened that the sum of the U's exceeds one by the unit of the last decimal place, which is due to the necessity of correcting the single values of the U's. The course of the increase and decrease of the U's is regular throughout the entire table. From these values of the U's one finds the products $r_k U_k$ and $r_k^2 U_k$, which are given in the following two tables (Tables 67 and 68, Appendix p. 206). The sum of the $r_k U_k$ is the determination of the just imperceptible negative difference. These values compare thus with the just perceptible negative difference:

Subject.	Just perceptible negative difference.	Just imperceptible negative difference.
I	93.0849	93.5089
II	95.2780	94.4648
III	97.4041	98.2963
IV	95.2281	95.5572
V	94.1966	94.7417
VI	95.5218	95.0922
VII	96.0379	95.5472 .

The just perceptible negative difference is smaller than the just imperceptible difference in four cases and larger in three cases. There is in our results no indication of a constant relation between the just perceptible and the just imperceptible negative difference. The data of the last two tables may be used for representing the results in this way. There exists a probability greater than $1 - \frac{t^2}{n}$ that the result of the observation will fall within the limits

$93.51 - \frac{4.46}{t}$	and	$93.51 + \frac{4.46}{t}$	for the subject	I;
$94.46 - \frac{4.71}{t}$	"	$94.46 + \frac{4.71}{t}$	" " "	II;
$98.30 - \frac{4.99}{t}$	"	$88.30 + \frac{4.99}{t}$	" " "	III;
$94.74 - \frac{5.67}{t}$	"	$94.74 + \frac{5.67}{t}$	" " "	IV;
$94.74 - \frac{4.19}{t}$	"	$94.74 + \frac{4.19}{t}$	" " "	V;
$95.09 - \frac{4.93}{t}$	"	$95.09 + \frac{4.93}{t}$	" " "	VI;
$95.55 - \frac{4.78}{t}$	"	$95.55 + \frac{4.78}{t}$	" " "	VII.

The coefficients which determine the limits within which we may expect the result are small, so that a close coincidence of the theoretical results with the observed results may be expected. The just imperceptible negative difference is the smallest stimulus of a series on which a judgment is given which is not a "lighter"-judgment. In the example of a record on p. 6, we have in the first column 84, in the second column 88, in the third column 96, in the fourth column 96 and in the fifth column 92 as determinations of the just imperceptible negative difference. The results of our experiments are given in Tables 69-75 (Appendix pp. 207-209, 211), in the same way as in the previous tables and the signification of the letters is also the same. The next table (Table 76, Appendix p. 210) gives the combined results for the determination of the just imperceptible negative difference. In order

to make the comparison between the theoretical and the observed results easier Table 77 (Appendix p. 211) was constructed which shows the coincidence of these quantities. The differences between the observed and the theoretical results in the different series of 100 experiments are small, and the differences between the theoretical results and the results of the combined series (see Table 77 under the heading "Totals") are so inconsiderable that they may be neglected for all practical purposes; the only exception is subject VII, for whom this difference is somewhat considerable. This is due to the fact that for this subject all the deviations from the most probable result are positive. This large difference between the observed and the theoretical values for subject VII may be looked at as just as much of a chance as the very close coincidence for the subjects III and VI. It is of importance to compare the results of the observations on the just imperceptible negative difference with those of the observations of the just perceptible negative difference. This serves also the purpose of getting material for answering the question whether it is an advantage to take the arithmetical mean of the just perceptible and of the just imperceptible difference.

Subject.	Just perceptible negative difference.	Just imperceptible negative difference.	Difference.
I	93.16	93.82	0.66
II	95.24	94.71	0.53
III	97.45	98.32	0.87
IV	95.05	96.07	1.02
V	94.22	94.91	0.69
VI	95.36	95.04	0.32
VII	96.04	97.44	1.40

The difference between these quantities* are small and in this case it will be a decided advantage to take the arithmetical mean as the representative value, as it is prescribed by the method of just perceptible differences. We shall make the following considerations on the method of determining the threshold as the arithmetical mean of the just perceptible and of the just imperceptible differences. There are only two cases possible: either

the stimulus for which there exists the probability $\frac{1}{2}$ that a "greater" or that a "smaller"-judgment will be given lies in the interval between the empirical determinations of the just perceptible and of the just imperceptible difference, or the empirical determinations lie both on the same side of this value. The arithmetical mean comes closer to the true value than at least one of the determinations in both cases. Add to this that it is very desirable to have the final results in as simple a form as possible, and there will remain little doubt that the combination of the just perceptible and of the just imperceptible difference is a very serviceable means for the determination of the threshold in the direction of increase and in the direction of decrease. These facts are independent of the theoretical consideration that the just perceptible and the just imperceptible differences are empirical determination of the same quantity, so that their average has the character of being the most probable result. The two following tables serve the purpose of showing the superior accuracy of the mean of the just perceptible and of the just imperceptible differences. Table 78 (Appendix p. 212 sq.) is a composite table of the individual results of all the series. The results for every subject are given in one group. The results of the observations on the just perceptible positive difference, on the just imperceptible positive difference, on the just perceptible negative difference and on the just imperceptible negative difference are given on one line for every series of 100 experiments. The theoretical values of all these quantities are given for every subject in the top line of the group. The column marked "Observed values" contains the results of the observations and the numbers opposite to these give the differences between the observed and the theoretical values. These differences are given the positive sign if the observed value is greater than the theoretical value, and the negative sign if it is smaller. The averages at the bottom of the columns refer to the differences taken regardless of sign. The data of Table 79 (Appendix p. 214) are arranged in a similar way, the only difference being that the numbers do not refer to just perceptible and just imperceptible differences, but to the averages of the just perceptible and just imperceptible difference, i. e. to the thresholds in the direction of increase and to those in the di-

rection of decrease. The comparison of the averages of the differences between the theoretical and the observed values, the differences being taken regardless of sign, with those of the preceding table shows, indeed, that at least one of the just perceptible and of the just imperceptible differences has a larger sum of the deviations of the observed from the calculated value than the corresponding threshold. The threshold in the direction of decrease, for instance, is the average of the just perceptible and of the just imperceptible negative difference. For subject III the deviations of the observed values from the theoretical value of the threshold in the direction of decrease are on the average 0.495 (Table 79), which is smaller than 0.580, the average of these deviations for the just imperceptible negative difference, but which is greater than 0.488 (Table 78), the average of the deviations for the just perceptible negative difference for the same subject. It happens in four cases (the positive thresholds for the subjects I, II, III and VII) that the sum of the deviations for the threshold is smaller than any one of those for the corresponding just perceptible and just imperceptible differences. It is easy to see that the probability of this event is equal to the probability that the theoretical value lies in the interval between the just perceptible and the just imperceptible difference, and that it is nearer to the arithmetical mean of the observations than to either one of them. The probability of either one of these events is $\frac{1}{2}$ and the probability of the compound events is, therefore, $\frac{1}{4}$. Our results show that this event happened in 4 out of 14 cases, which is exactly what we have to expect. This coincidence is one more argument that the conditions of randomness of events are very closely realized in our experiments.

The final evaluation of the results of the method of just perceptible differences is given in the Tables 80 and 81 (Appendix p. 215). The just perceptible and the just imperceptible differences are combined to find the threshold in the direction of increase and decrease. The results are given up to the second decimal place. Table 80 gives the theoretical results and the following table gives the observed results. The coincidence of the theoretical with the observed results is very close for all the subjects except for subject VII, for whom it is less satisfactory. The

two thresholds define an interval outside of which there exists a probability exceeding $\frac{1}{2}$ either for a "heavier" or for a "lighter"-judgment. We may express this fact shortly by saying that for differences beyond the threshold there exists a probability exceeding $\frac{1}{2}$ that they will be recognized. Inside of this interval one can not expect either a "heavier" or a "lighter"-judgment with an even or more than even probability, and for this reason it is, perhaps, appropriate to call this interval the interval of uncertainty. We give here a comparison of the theoretical and of the observed length of this interval.

Subject.	Theoretical value of interval of uncertainty.	Observed value of interval of uncertainty.	Difference.
I	6.15	6.11	-0.04
II	3.96	3.74	-0.22
III	1.43	1.70	+0.27
IV	2.69	2.68	-0.01
V	2.67	2.78	+0.09
VI	4.56	5.12	+0.56
VII	4.07	2.88	-1.19.

The observed values exceed the theoretical values in three cases, and they fall short of them in four cases. The average of all the deviations (taken regardless of sign) is 0.34. It is important to have an idea of the accuracy of the determination of this quantity, because the interval of uncertainty is the basis of the psychophysical measurements afforded by the method of just perceptible differences, as it is used to-day. Indeed, let us call the intensity of the stimulus which corresponds to subjective equality r , the just perceptible positive difference r'_0 , and the just imperceptible positive difference r''_0 . We then have

$$r_0 = \frac{1}{2} (r'_0 + r''_0)$$

and

$$\Delta r_0 = r_0 - r.$$

Calling the just perceptible negative difference r'_u and the just imperceptible difference r''_u we find in the same way

$$r_u = \frac{1}{2} (r'_u + r''_u)$$

and

$$\Delta r_u = r - r_u.$$

The quantities Δr_u and Δr_o are combined as a rule into an average, which is called the mean threshold of difference and which is defined by

$$\Delta r = \frac{1}{2} (\Delta r_u + \Delta r_o).$$

Introducing the values r , r_u , r_o we obtain

$$\Delta r = \frac{1}{2} (r - r_u + r_o - r) = \frac{1}{2} (r_o - r_u)$$

and we see from this formula that the mean threshold of difference is nothing else but half the length of the interval of uncertainty. The method of just perceptible differences prescribes to take this quantity as a measurement of the accuracy of sensation, putting this accuracy inversely proportional to the threshold of difference.

It is of interest to notice that the value r has dropped out from the formula for the final result of the method of just perceptible differences. It is, therefore, not necessary to determine it and, as a matter of fact, special care must be devoted to the determination of the point of subjective equality in those investigations where only the threshold in the direction of decrease or the one in the direction of increase is determined, or where the amount of overestimation or underestimation is of interest. Observations on the point of subjective equality are only an incident in the method of just perceptible differences. There is nothing in the data of this method which may serve as a definition of the objective difference of stimuli which corresponds to subjective equality, and it is quite impossible that this method gives any indications about it, because its algorithm entirely disregards the "equality"-judgments and a judgment on the difference which corresponds to subjective equality can hardly be based on anything else but on the "equality"-judgments.

Our discussions show that the result of the experimental procedure which is called the method of just perceptible differences is expressible in terms of probabilities of judgments belonging to certain types, and that the method even gains in lucidity by doing so. These judgments are, of course, either correct or incorrect and there does not exist a fundamental difference between this method and the error methods in so far as both use the same data. There may be, however, a very striking difference in so far as the attitude of the subject is concerned. The results of every psychological experiment may be worked out in two ways. In the first case all one is after are statistical numbers of relative frequency. The introspections of the subject are of little interest and they are of importance only in so far as they control the correct performance of the experiments. The only statement which the subject has to make in regard to his mental states is: "I complied with the conditions of the experiment." The subject is regarded as an instrument which automatically shows when its function is not such as it ought to be. The experimental results which one obtains serve for the determination of those factors which are of influence on the function of the subject and the analysis of this function is the only purpose of the investigation. The second way of performing psychological experiments is to lay stress primarily on the introspections of the subject. The numbers of relative frequency which are gained in such experiments are of interest merely in so far as they give an indication as to the constance or variability of the conditions under which a certain psychical state arises. It may seem advisable to characterize the first way of experimenting as the psychophysical attitude, and the second as the psychological attitude in the restricted sense of the word. Both are of equal importance and neither one must be neglected at the expense of the other, but confusion is inevitable if they are not strictly separated. If one uses the method of just perceptible differences as a purely psychological method the only purpose of the experimental arrangement is to help the subject in finding a stimulus on which he may give the judgment that it is the smallest difference which he can perceive. The observations which are given alongside with this judgment or in the course of reaching

this judgment are by far more important than the numerical value of the observed difference. The variations of the single observations are, perhaps, more interesting than the final result, because they give an indication of the variability of the mental state under observation. The purpose of the experiments is absolutely different if one uses the method of just perceptible differences as a purely psychophysical method. In this case one wants to form one's view about the process of judging differences by means of the observed numbers of relative frequency. These numbers are the only outcome of the experimentation and they must be put to the best possible use, since it must be required that all is got out of the results what is in them. The error-methods are used primarily as psychophysical methods, whereas the method of just perceptible differences may be easily adapted for both purposes.

The notion of a just perceptible difference was the object of frequent attacks which were met by an equally stubborn defence, because this notion seemed to be the indispensable basis of quantitative psychology. Fechner, who utilized this notion for this purpose, had given it the signification of an increment of sensation thus inaugurating the view that a sensation is the sum of very small sensations. This view in connection with Weber's observation of the constance of the relative threshold leads to the establishment of Fechner's fundamental formula for psychical measurement. This theory combines in a peculiar way the features of the psychophysical and of the psychological attitude. The notion of a difference which the subject just perceives, indeed, is a purely psychological one, and it cannot possibly be found in the statistical numbers of relative frequency, unless it is deprived of its psychological character and transformed into the notion of a difference for which there exists the probability $\frac{1}{2}$ that it will be recognized. There is no objection against the view that a sensation is constituted of elementary sensations from the psychophysical point of view, because one understands in this case by sensation nothing but a certain process, which may very well be the sum of other processes. The question merely is whether this hypothesis is serviceable, but the objection that a sensation is given as a whole and not as the sum of smaller sensations is

very obvious and very decisive for the psychological point of view. This objection was raised very soon against Fechner's formula, but there were on the other hand certain physiological facts concerning the theory of sensations which spoke in favor of this view. The possibility of composing sensations of their elements seemed to give the possibility of exhausting the manifoldness of sensations by a limited number of simple sensations. For these simple sensations one might postulate different physiological processes thus reducing the study of sensations to that of a limited number of elementary processes. The principle of specific sense energy, thus, became more or less consciously a factor in favor of Fechner's view. At the same time the notion that a sensation is the sum of elementary sensations opened up the way for the application of the theory of errors. The sensation depends only on the combination of elementary sensations and it depends on chance which combination is aroused by a certain stimulus. The limits inside of which the result may be expected with a given probability determine the accuracy or precision of a sensation.* On this basis one may try find the probability of correct judgments for given differences of the intensities of two stimuli. The way in which the solution of this problem was attempted may be illustrated by the following example. There are two rifles, A and B, in a fixed position in such a way that A is aimed at a point which is somewhat higher than the point at which B is aimed. This difference is known. These rifles are fired a great number of times and one observes in each case whether the bullet of the rifle A struck a higher point than that of the rifle B. It is required to form a judgment on the precision of the rifles on the basis of these data. The solution of

*The expression, "*precision of sensation*" is taken from ROBERT MUELLER *Ueber die Grundlagen der Richtigkeit der Sinnesaussage*, *Journal f. Psychologie u. Neurologie*, Vol. III, 1904, pp. 112-126, but essentially this notion was used already by FULLERTON and CATTELL *l. c.* p. 12, "Owing to the complex physical, physiological and psychological antecedents of the perception, the same stimulus is not always accompanied by the same sensation. There is a normal error of observation" These authors introduce in the further progress of their discussion the hypothesis that this normal error of observation is the sum of elementary errors, a hypothesis which leads of course to the probability integral.

this problem requires the $\Phi(r)$ -function and one obtains formulæ which are analogous to those used in the method of right and wrong cases.

The agreement with the theory of specific sense energy and the possibility of obtaining a safe starting point for the quantitative methods in psychology were strong arguments in favor of the view that a sensation is built up from sensational elements. Our considerations show that it is not necessary to conceive of the result of the method of just perceptible differences as determining a sensational element, and they leave to this difference merely the dignity of possessing a distinguished numerical value of the probability with which a certain class of judgments is passed. Our analysis shows that it is not necessary to base the theory of psychophysical measurement on this physiological theory, a blow which at the present time is more serious for this theory than for the deductions of quantitative psychology.

We see that very diverse interests came into play in the problem of Fechner's fundamental formula. In this formula psychophysical and psychological views are intermingled in a very peculiar way, and it seems that this combination of view, points which are essentially different, is the source of the difficulties which arise in the interpretation of this formula. It is indispensable to separate the psychological and the psychophysical attitude strictly and to work out both consistently. The statistical numbers of relative frequency of the different judgments are immediately given and accessible to everybody. The sensations of the subject are not known to anybody but to himself, and we have to disregard them entirely for the purpose of an objective treatment of the results in the same way as we disregard the inner states of a body with which we deal in physics. These considerations show clearly that we do not measure sensations in psychological experiments, neither do we determine the relations of sensations to certain differences of stimuli. What we determine are the probabilities of certain events and the nature of their dependence on certain physical conditions, just in the same way as we do not measure heat but certain linear magnitudes. This does not interfere with our being able to solve all the problems of psychophysics just in the same way as if we could measure sensations. All

the problems which are treated in quantitative psychology are accessible either by one of the error methods or by the method of just perceptible differences, or by both. We have seen that the result of the method of just perceptible differences is expressible in terms of probabilities, and so are the results of the error methods. It follows from this that every problem which can be treated in psychology at the present time is expressible in terms of probabilities of the different judgments.

CHAPTER IV.

THE EQUALITY CASES.

In our experiments the cases in which a "guess"-judgment was given are equivalent to equality cases. The instructions given to the subject call for a "guess"-judgment when no difference between the two stimuli could be detected; there ought to be nothing in the sensation to go upon in the formation of the judgment that either one of the weights is heavier. It is of some interest to study the number of times the second weight was guessed to be lighter or heavier when the subject believed to be unable to detect a difference. We will study the distribution of the guesses between the "hg" and "lg"-judgments by means of the ratio of the number of "hg"-judgments to the number of "lg"-judgments, because only the relative frequency is of interest, and because this ratio allows us to express the results by one number, thus reducing the size of the table by one-half. The necessary data are given in Tables 3-9 (Appendix pp. 174-177). These ratios are given in Table 82 (Appendix p. 215) for all the subjects except subject III for each pair of comparison weights separately. Numbers of one column refer to the same subject and numbers in one line refer to the same pair of comparison weights. The results for subjects III are not given, because this subject did not succeed in conforming himself to the conditions of the experiments in so far as the use of the guesses was concerned. This subject gave very few "guess"-judgments, and those which he gave were almost exclusively "hg"-judgments. An inspection of Table 82 shows that the numbers in the upper third of the table are, generally speaking, smaller than 1 and that they are in the lower part of the table greater than 1 without a single exception. This means that for negative differences the "lighter-guesses" are in the majority and that for positive differences considerably more "heavier-guesses" are given.

biometrical sciences. The values most frequently chosen as representative for the entire set of results are the median, the mode and the arithmetical mean. It is obvious that the median can not be used in our case, because the range of the differences used is determined not by objective conditions but by the chance element of subjective choice. It would be, perhaps, different if this range were so wide that the proximal and the distal terms of the table of distribution had a frequency equal to or near to zero. The determination of the mode meets with considerable algebraical difficulties and its position does not throw much light on this problem as will be seen in the following chapter. There remains, therefore, only the arithmetical mean to be considered here.

Let us make, for the purpose of this discussion, the following considerations. Suppose we make a determination of the standard weight with a very crude balance. The set of weights with which we have to compare the standard weight of unknown intensity may consist of the weights 84, 88, 92, 96, 100, 104, and 108 gr. We make with this instrument as many comparisons of every weight with the standard as we make experiments with a subject. Our balance will show in some cases that the standard weight is heavier than the comparison weight, in other cases it will be found to be lighter and, since our instrument is very inaccurate, we will not be able to detect any difference at all in a certain number of cases. Those cases where no difference can be detected are noted as determinations of the weight of the standard, and the discrepancies between the single observations will have to be eliminated by the method of least squares. Making these determinations with different instruments and known intensities of weight we shall obtain results which not only allow us to make a statement about the comparative accuracy of the balances, but which also enable us to tell whether our instruments are affected by a constant error. Disregarding entirely the "heavier" and "lighter"-judgments we suppose that our instrument has shown the standard to be equal to every one of the weights as many times as our subject could not detect a difference between the weights. The total number of experiments is the same in both cases.

A set of results arranged for this purpose is given in Table 84 (Appendix p. 216). The columns N_k give the number of times each stimulus seemed to be equal to the standard, i.e. was obtained as a determination of its weight. These results must be regarded as determinations of different pondus* of the same weight and the column under the heading $N_k r_k$ gives the number with which each observation comes down for the determination of the arithmetical mean. The sums of these numbers are given at the bottom of the columns in the line marked Σ . Dividing the sum of the $N_k r_k$ by the sum of the N_k one obtains the most probable value for the determination of the weight of the standard; these numbers are given for each subject in the line marked "Average". One then proceeds to find in the usual way the deviations of the observations from the mean and the sum of their weighted squares obtaining the final result in the form:

I	96.10 $\pm$ 1.44
II	96.43 $\pm$ 1.68
III	97.78 $\pm$ 1.36
IV	97.15 $\pm$ 1.52
V	96.98 $\pm$ 1.38
VI	97.65 $\pm$ 1.42
VII	98.03 $\pm$ 1.58.

We would not hesitate to say that instruments which give such results are not only very inaccurate, but that they are also affected by a considerable constant error. The difference of the general mean from 100 gr would have to be taken as the constant error which in all our cases is negative, so that we have clearly to deal only with subjects who overestimate the second weight. The comparison of the accuracy can not be based directly on the numbers given above for the probable error of the mean, because

*The use of the English term "weight" would necessitate constant circumlocutions and involve the possibility of a mistake, because this term would be used in two different meanings. It seemed appropriate for this reason to use the Latin term whenever the word "weight" had to be used in the meaning which it has in the theory of least squares.

the different series are not equally extended, so that the determinations for the different subjects have not the same pondus. The relative pondera may be found by the following consideration. It was necessary to make 3150 experiments with subject III in order to obtain 168 determinations of the weight of the standard, whereas with subject VI only 2100 experiments were needed to obtain 401 determinations. The pondus which we give to the different determinations depends on the amount of work required for obtaining the results, and since all the experiments were made with equal care, we put the pondus proportional to the number of experiments performed. If the pondus of the determinations for the first three subjects is 1, that for the subjects IV-VII will be only $\frac{2}{3}$. It is indifferent whether we base our comparison of the accuracy of the determinations of the weight of the standard by the different subjects on the coefficient of precision or on any other quantity which is derived from it. We choose the probable error of a single determination, this being directly comprobable for all the subjects. The values of the probable errors of a single determination for the different subjects are given here along with the length of the interval of uncertainty determined above by the method of just perceptible differences.

Subject.	Probable Error.	Interval of Uncertainty.
I	38.37	6.15
II	35.79	3.96
III	17.69	1.70
IV	26.69	2.69
V	24.07	2.67
VI	34.82	4.56
VII	38.05	4.07.

These results show that in general a large probable error and a large interval of uncertainty are found in the same subject. This correlation will become clearer if we arrange our seven subjects first by the size of their probable errors and then by the length of their intervals of uncertainty. Beginning with the subject who has the largest probable error and with the one who has the

greatest interval of uncertainty we obtain the following two arrangements:

Order of the subjects by probable error.	Order of the subjects by interval of uncertainty.
I	I
VII	VI
II	VII
VI	II
IV	IV
V	V
III	III.

This little table shows that these two orders go parallel to a certain extent. The subject who appears first in one order has also the first place in the second order, and the subject who is last in one order has the same place in the other order. There are, furthermore, two subjects (IV and V) who appear at the same places in both arrangements. The other three subjects appear at different places in the two arrangements, but subjects II and VII appear in the same succession, so that in so far as the succession is concerned only subject VI is out of his place. Subject VII is on second place and subject II is on third place in the order by the size of the probable error, and these subjects have third and fourth place in the order by the length of the interval of uncertainty. This means that only one out of seven subjects appears out of the order of succession in the two arrangements. We may express this result by saying that the size of the probable error in general goes parallel to the length of the interval of uncertainty.

The calculation of the probable error is based on the sum of the weighted squares of the deviations from the arithmetical mean. G. Fr. Lipps based a calculation of the threshold on the same value, and he found that the square of the quantity which he calls the threshold must be smaller than three times the sum of the squares of the deviations. Our results bear out in a general way that this quantity may be used as a measurement of sensations, because the order of the subjects when

they are arranged by the size of this quantity will be similar to the one if the subjects are arranged by the length of the interval of uncertainty. The same holds good for any other measurement of sensations which is based on the sum of the squares of the deviations multiplied by a constant factor.

CHAPTER V.

THE PSYCHOMETRIC FUNCTIONS.

Until now we treated the probabilities of the judgments for various amounts of difference in the intensities of the stimuli as unrelated quantities. It is a very natural supposition that these probabilities depend on the amount of difference between the standard and the comparison stimulus. We immediately give to this supposition the form in which it is useful for further treatment by assuming that these probabilities are analytic functions of the difference between the stimuli.* There exist, therefore, three different functions one of which expresses the probabilities of "lighter"-judgments, the second the probabilities of "heavier"-judgments and the third the probabilities of "equality"-judgments as depending on the amount of difference between the stimuli. This is, indeed, nothing but a formulation of the fact that the probabilities must depend in some way on the amount of difference between the stimuli. We have to expect for large negative differences large percentages of "lighter"-judgments and small percentages of "equality"-judgments and of "heavier"-judgments. In a similar way we expect with a high probability "heavier"-judgments for considerable positive differences, for which differences the probabilities of "equality"-judgments and of "heavier"-judgments are small. We make no definite hypothesis about the nature of this dependence but we merely try to see how much one can say about the results with the minimum addition of theoretical suppositions. The functions which give the dependence of the probabilities of the judgments on the differences of the stimuli may be called the functions

*This supposition has not the character of a specific hypothesis, but that of a general supposition necessary for mathematical treatment. A discussion of the importance of this far reaching assumption is given in the last chapter.

of the distribution of these judgments, or in view of the importance which they have for the problems of quantitative psychology one may call them the psychometric functions. This term is an imitation of the term "biometric function" which is in common use for functions which represent the mortality for the different ages. There are as many psychometric functions as one admits types of judgments. In our experiments we have to distinguish the psychometric function for the "heavier", for the "lighter" and for the "guess"-judgments. The number of psychometric functions would be increased considerably if the degree of subjective confidence were taken in consideration.

Our problem, then, is formulated in this way. The values of a function were observed for a series of differences and the results must serve for the determination of the function; what is the course of this function and what are its values for other differences? The answer is relatively simple if the nature of the function is known. Such a function, indeed, must depend on a certain number of parameters, which can be directly determined if the number of observations is equal to their number, and the most probable values of which can be determined by the method of least squares if the number of observations exceeds the number of the constants of the function. If the number of observations is smaller than the number of parameters, the problem is undetermined. If the function is not known, one finds the values for other differences by interpolation. The interpolation may be effected either by means of Newton's method of differences or by Lagrange's formula of interpolation. Both methods are based on the assumption that a set of n observations can be represented by an algebraic function of degree $(n-1)$, but even if this supposition is not justified (the function being of higher than the $(n-1)^{\text{th}}$ degree), one may take the values given by these formulæ as values of the simplest function by which one can account for the n different values.*

*The term "the simplest function" is due to GAUSS, *Theoria interpolationis methodo nova tractata*, Werke, Vol. 3, p. 275 "Quae praecedunt suppositione innituntur, functionem X ultra potestatem x^{m-1} non egredi, sive differentiam m^{tam} una cum superioribus evanescere, in quo casu methodus interpolationis rigore vera est. Si vero illa suppositio locum non habet,

The results of our experiments when brought into the form in which they are given in Table 85 (Appendix p. 217) represent sets of seven observations on the three psychometric functions for each one of our seven subjects. The observations are made for the standard weight of 100 gr and with the seven comparison weights of 84, 88, 92, 96, 100, 104, and 108 gr. We call the results of these observations $a_1, a_2, a_3, a_4, a_5, a_6,$ and a_7 . The table gives these values for all the subjects under the headings G, H and L, so that we have the data for the determination of all the psychometric functions. The general expression for Lagrange's formula is

$$F(x) = \frac{(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}a_1 + \frac{(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}a_2 + \dots \\ + \frac{(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})}a_n$$

where $x_1, x_2, \dots, x_n$ are the values for which the observations were made and $a_1, a_2, \dots, a_n$ are the results of these observations. Introducing the data of our experiments in this formula we obtain

$$F(x) = \frac{(x-88)(x-92)(x-96)(x-100)(x-104)(x-108)}{4 \cdot 8 \cdot 12 \cdot 16 \cdot 20 \cdot 24}a_1 - \\ - \frac{(x-84)(x-92)(x-96)(x-100)(x-104)(x-108)}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 16 \cdot 20}a_2 + \\ + \frac{(x-84)(x-88)(x-96)(x-100)(x-104)(x-108)}{8 \cdot 4 \cdot 4 \cdot 8 \cdot 12 \cdot 16}a_3 - \\ - \frac{(x-84)(x-88)(x-92)(x-100)(x-104)(x-108)}{12 \cdot 8 \cdot 4 \cdot 4 \cdot 8 \cdot 12}a_4 + \\ + \frac{(x-84)(x-88)(x-92)(x-96)(x-104)(x-108)}{16 \cdot 12 \cdot 8 \cdot 4 \cdot 4 \cdot 8}a_5 - \\ - \frac{(x-84)(x-88)(x-92)(x-96)(x-100)(x-108)}{20 \cdot 16 \cdot 12 \cdot 8 \cdot 4 \cdot 4}a_6 + \\ + \frac{(x-84)(x-88)(x-92)(x-96)(x-100)(x-104)}{24 \cdot 20 \cdot 16 \cdot 12 \cdot 8 \cdot 4}a_7.$$

interpolatio eo tantummodo tendit, ut loco functionis X *functio simplicissim* eruatur, per quam valoribus $A, B, C, D, \dots$ satisfiat."

This equation of sixth degree assumes the value a_1 for $x=84$, the value a_2 for $x=88$, ...and the value a_7 for $x=108$. Introducing for x any particular value one obtains the corresponding value of the psychometric function. It may be allowed to make the following remarks in regard to the theoretical and practical application of this formula. It is at once obvious that this equation can not possibly represent the psychometric function in its entire course. This function, indeed, is by its nature of representing a mathematical probability limited to values between zero and one, the limits included. Values greater than the unit and smaller than zero can not be admitted. An algebraic function has the limit ∞ for $x = \infty$, i. e. it assumes very great values for considerable values of its variable and one, therefore, must not undertake to extrapolate by this formula. The occurrence of values greater than the unit and of negative values inside of the realm of the observations has only symptomatic significance.

The practical application of Lagrange's formula is extremely simple and easy although a little laborious. It is a matter of course that one must arrange the computation systematically whenever one has to treat an experimental material which is somewhat extended. In this respect it is of importance that the a 's are the same throughout the whole computation for a psychometric function of one subject. The interpolation for different values of x will be effected with the same values of the a 's but with different values of the coefficients. The values of these coefficients, on the other hand, do not depend on the a 's and remain the same for different subjects. It is, therefore, possible to arrange the computation in two ways, if one has to treat the results for different subjects with whom the same differences were used. The first way is to attack the results for all the subjects at once, making the interpolation for one value of x for all the subjects at the same time. The other way is to carry out all the interpolations for one subject before beginning those for another. Every interpolated value of our tables is the result of seven multiplications, two additions and one subtraction. This work is reduced considerably by the logarithmic computation which consists almost of nothing else but copying logarithms, which were looked up once

for all, and performing simple additions, if the computation is arranged appropriately. The only inconvenience of this work is that there are no thoroughgoing checks, so that the calculation must simply be done over again. The results of this computation are shown in the Tables 86-92 (Appendix pp. 218-221). The values of the psychometric functions for the "equality" and "heavier"-judgments were obtained by direct interpolation and the values for the "lighter"-judgments were found by subtraction from one. The tables show the values of the psychometric functions for the "heavier", for the "lighter" and for the "equality"-judgments for the values between 84 and 108 gr with intervals of 1 gr, so that three values are interpolated between two observations. The column under the heading G gives the values of the psychometric function for the "equality"-judgments, and the letters H and L at the head of the third and second column stand in abbreviation of the psychometric function for the "heavier" and of that for the "lighter"-judgments. The general trend of these numbers conforms to our expectation about the course of the psychometric functions. The numbers in the column G set in with low values and gradually rise up to a certain maximum, and after this is attained they decrease again. The numbers H set in with very low values and increase at first slowly then more rapidly assuming very high values for positive differences. The mere inspection of the tables shows two striking features. The first is the occurrence of negative values and of values greater than the unit in the results of all the subjects except for the subjects IV and VI. These breaks occur at the extreme ends of the tables. The second remarkable feature is the smallness of the values in the column G. These numbers never come anywhere near the unit and only for subject I do they approach the value $\frac{1}{2}$. For none of the other subjects do they exceed the value 0.4 and for two subjects they do not reach the value 0.2 (subjects III and V). This confirms our previous observation that one can not speak in any absolute sense of the word of a point of subjective equality, because for this value one would have to require an at least even probability for the "equality"-judgments. There remain still two possibilities of defining the point of subjective equality. In the first case the probability

of an equality judgment is greater than the probability of either a "greater" or a "smaller"-judgment although it remains smaller than $\frac{1}{2}$ and, therefore, smaller than the sum of the probabilities of the "greater" and "smaller"-judgments. The interval of subjective equality is represented in this case by those differences in the intensities of the stimuli for which the outcome of a comparison is more likely to be an "equality"-judgment than either a "greater" or a "smaller"-judgment, but for which the equality judgments are less frequent than the two classes together. There still remains the difficulty of defining the point of subjective equality, because one has the choice between the centre of the interval of subjective equality and the value for which the probability of an "equality"-judgment is a maximum. An interval of subjective equality exists only if there is an interval inside of which the psychometric function of the "equality"-judgments rises above the psychometric functions of the "greater" and of the "smaller"-judgments. If there is no such interval one must define the point of subjective equality as the value for which the psychometric function of the "equality"-judgments attains its maximum. This definition does not imply that "equality"-judgments are more probable than "greater" or "smaller"-judgments, but it merely says that for this point the judgments of equality are more frequent than for any other difference of the stimuli. It is not necessary that there exists an interval of subjective equality in which the psychometric function of the "equality"-judgments rises above the psychometric functions of the "greater" or "smaller"-judgments. This definition of the point of subjective equality is the most general, because it does not make any assumption about the course of the psychometric functions.

The further analysis of the results of our experiments is considerably facilitated by the graphical representation of the psychometric functions as given in the Charts I-VII. On the lines XX' of the charts the intensities of the comparison stimuli are represented, the unit of the linear measurement corresponding to 1 gr. The ordinates represent the probabilities of the different judgments, the unit of the Y axis being chosen ten times as large as that of the abscissa. The lines NN' are drawn in the unit of

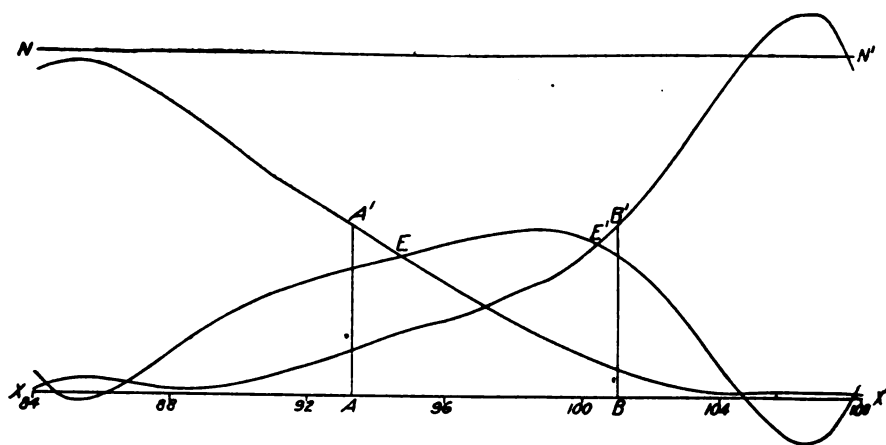


CHART I.

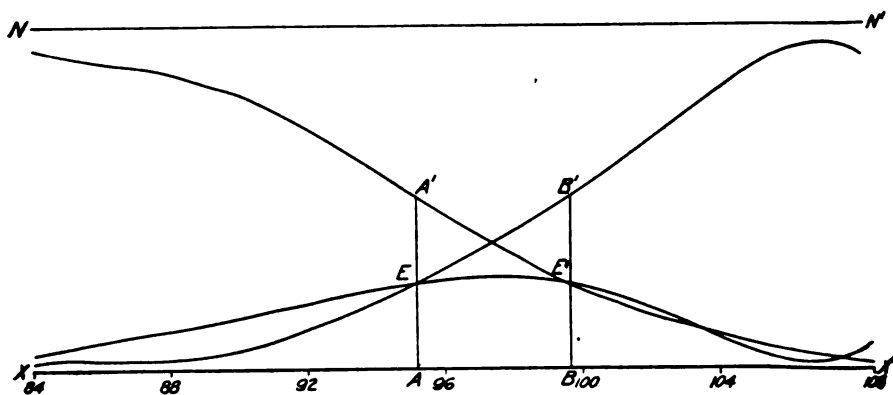


CHART II.

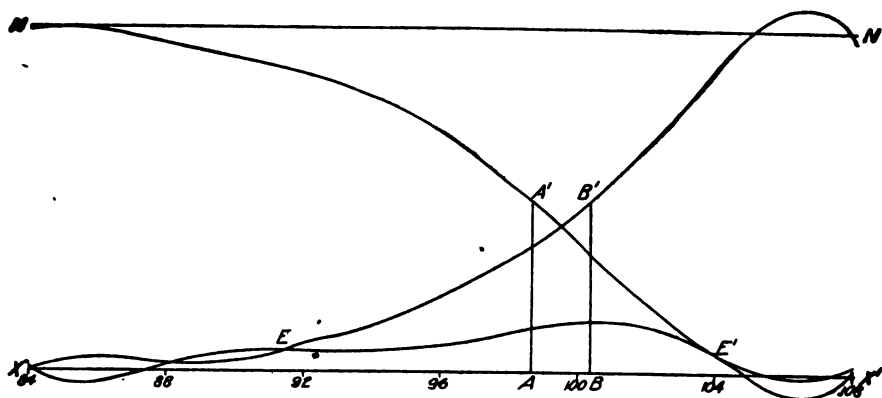


CHART III.

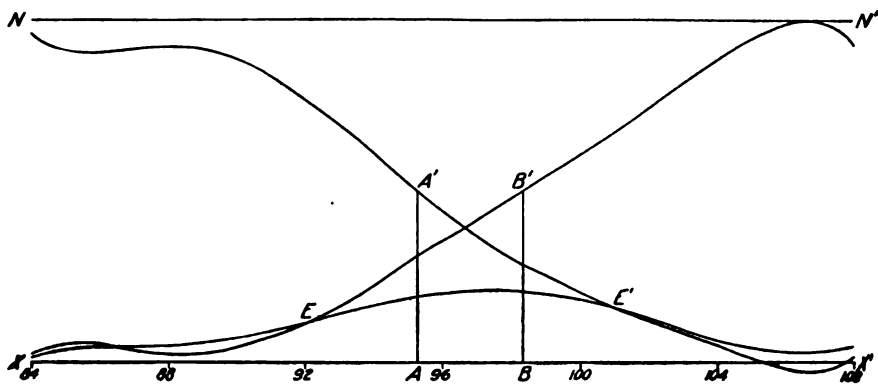


CHART IV.

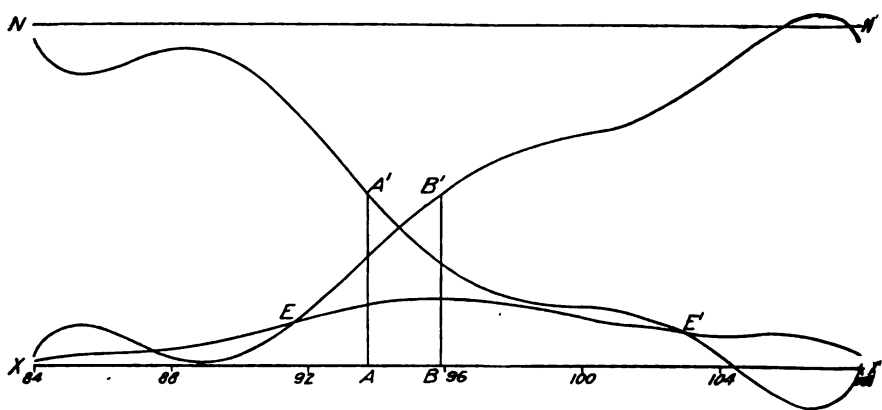


CHART V.

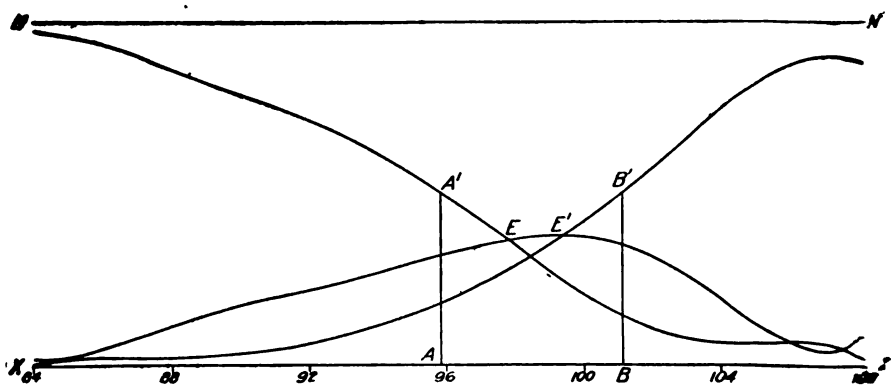


CHART VI.

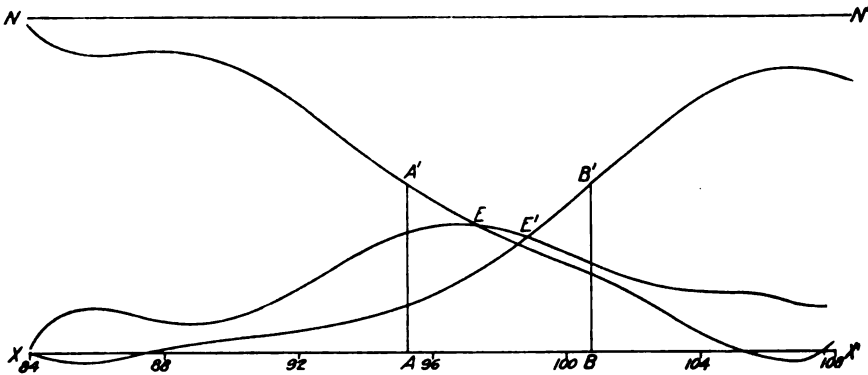


CHART VII.

distance from XX' . The psychometric function for the "lighter"-judgments are represented in each chart by the curve passing through the point A' ; the psychometric function for the "heavier"-judgments is represented by the curve passing through B' , and the curve passing through the points E and E' represents the course of the psychometric function for the "equality"-judgments. The point A marks the values of x for which the psychometric function for the "lighter"-judgments assumes the value $\frac{1}{2}$; A' is the corresponding point on the curve. The point B gives the intensity for which the psychometric function for the "heavier" judgments assumes the value $\frac{1}{2}$; B' is the corresponding point on the curve. The lines AA' and BB' have the length $\frac{1}{2}$. The general course of the curves is the same for all subjects. The tracings show at once whether the functions assume negative values or values greater than the unit, because in these cases the curves cross the lines XX' or NN' . We see, furthermore, that the course of the curves is in most cases rather irregular, increase alternating with decrease; only for the subjects II, IV and VI does it come near to regularity. There is, however, for all the subjects an interval inside of which all the three psychometric functions behave regularly and this interval lies approximately in the middle of the charts. We conclude from this fact that in our experiments the conditions are not quite the same for large and for small differences. This conclusion is in per-

fect agreement with introspective evidence and with the observation of previous investigators that the influence of attention in series of comparison stimuli which contain only small differences is different from that in series which contain large differences intermingled with small ones. The points for which the psychometric functions for the "heavier" and those for the "lighter"-judgments assume the value $\frac{1}{2}$ are always inside the interval in which these functions are regular. A close inspection of the curves in these intervals shows that the psychometric functions for the "heavier" and for the "lighter"-judgments differ only very slightly from straight lines. Perry's "test of the stretched thread" discloses scarcely any difference of the curves from straight lines in these intervals.

This circumstance suggests the following very handy determination of the value for which the psychometric functions for the "heavier" or for the "lighter"-judgments assume the value $\frac{1}{2}$, values of which we know that they have the signification of being the most probable result of the threshold in the direction of increase or of decrease as determined by the method of just perceptible differences. The complete solution of this problem would require to solve the equation $F(x) = \frac{1}{2}$. The work of solving this equation by one of the methods of approximation is insignificant when compared with the amount of work spent in setting up the equation by the formula of Lagrange, because the actual setting up of the equation requires a great number of long multiplications in which it is difficult to avoid mistakes. It is a very fortunate circumstance that the conditions for finding the approximate value for which $p = \frac{1}{2}$ are such, that we can avoid this long computation. One sees at once from the original data given in Table 85, in which interval the value in question must be sought. One includes the value for which $p = \frac{1}{2}$ in narrower and narrower limits by successive interpolations, until the interval is small enough to interpolate on a straight line. The charts show that it is not necessary to go in our results beyond intervals of 1 gr. In order to show the handiness of this method we will illustrate the course of this computation by finding the just perceptible positive difference for subject V from the data of Table 85 without having recurrence to those of Table 90, which

contains the values of the psychometric functions for this subject. The data of Table 85 show that the difference for which there exists the probability $\frac{1}{2}$ for a "heavier"-judgment must lie somewhere between 92 and 96. The numbers of relative frequency of the "heavier"-judgments (0.1600 for 92 and 0.5133 for 96) suggest that this value will be found in the upper part of the interval nearer to 96 than to 94. We interpolate by Lagrange's formula the value for 95. Introducing $x = 95$ into the formula given above on p. 108 we find the following values of the coefficients:

$$\frac{7. \ 3.-1.-5.-9.-13}{4. \ 8. \ 12. \ 16. \ 20. \ 24} = \frac{273}{65536} \text{ is the coefficient of } a_1 = 0.0267;$$

$$\frac{11. \ 3.-1.-5.-9.-13}{4. \ 4. \ 8. \ 12. \ 16. \ 20} = \frac{1287}{32768} \text{ " " " } a_2 = 0.0233;$$

$$\frac{11. \ 7.-1.-5.-9.-13}{8. \ 4. \ 4. \ 8. \ 12. \ 16} = \frac{15015}{65536} \text{ " " " } a_3 = 0.1600;$$

$$\frac{11. \ 7. \ 3.-5.-9.-13}{12. \ 8. \ 4. \ 4. \ 8. \ 12} = -\frac{15015}{16384} \text{ " " " } a_4 = 0.5133;$$

$$\frac{11. \ 7. \ 3.-1.-9.-13}{16. \ 12. \ 8. \ 4. \ 4. \ 8} = -\frac{9009}{65536} \text{ " " " } a_5 = 0.6800;$$

$$\frac{11. \ 7. \ 3.-1.-5.-13}{20. \ 16. \ 12. \ 8. \ 4. \ 4} = -\frac{1001}{32768} \text{ " " " } a_6 = 0.8700;$$

$$\frac{11. \ 7. \ 3.-1.-5.-9}{24. \ 20. \ 16. \ 12. \ 8. \ 4} = -\frac{231}{65536} \text{ " " " } a_7 = 0.9433.$$

It is, perhaps, worth while noticing that all the denominators are powers of 2. This is a consequence of our stimuli being chosen equidistant in intervals of 4 gr. This apparently trivial circumstance facilitates the computation not inconsiderably, and it is generally recommendable to use intervals which are powers of 2. In some cases it may be more convenient to make use of the method of differences for effecting the interpolation and if

one has to deal with equidistant values distributed over an interval which is a power of 2 one can apply the very handy formula for the "interpolation in the middle."

We begin conveniently by forming the sum of the negative terms, which are in this case the second, fifth and seventh term. The logarithmic computation may be arranged thus:

$$\begin{array}{rcl}
 \log 0.0233 & = & 0.36736 - 2 \\
 \log 1287 & = & 3.10958 \\
 \hline
 \text{Sum} & = & 1.47694 \\
 \log 32768 & = & 4.51545 \\
 \hline
 \text{Difference} & = & 0.96149 - 4 = \log 0.00092 \\
 \\
 \log 0.6800 & = & 0.83251 - 1 \\
 \log 9009 & = & 3.95468 \\
 \hline
 \text{Sum} & = & 3.78719 \\
 \log 65536 & = & 4.81648 \\
 \hline
 \text{Difference} & = & 0.97071 - 2 = \log 0.09348 \\
 \\
 \log 0.9433 & = & 0.97465 - 1 \\
 \log 231 & = & 2.36361 \\
 \hline
 \text{Sum} & = & 2.33826 \\
 \log 65536 & = & 4.81648 \\
 \hline
 \text{Difference} & = & 0.52178 - 3 = \log 0.00332.
 \end{array}$$

These are the three negative terms; their sum is

$$\begin{array}{rcl}
 & & 0.00092 \\
 & & 0.09348 \\
 & & 0.00332 \\
 \hline
 S_1 & = & 0.09772
 \end{array}$$

By a similar computation one may find the positive terms and form their sum. For this purpose one has to form the products for the first, third, fourth and sixth term.

$$\begin{array}{rcl} \log 0.0267 & = & 0.42651 - 2 \\ \log 273 & = & 2.43616 \end{array}$$

$$\begin{array}{rcl} \text{Sum} & = & 0.86267 \\ \log 65536 & = & 4.81648 \end{array}$$

$$\text{Difference} = 0.04619 - 4 = \log 0.00011$$

$$\begin{array}{rcl} \log 0.1600 & = & 0.20412 - 1 \\ \log 15015 & = & 4.17653 \end{array}$$

$$\begin{array}{rcl} \text{Sum} & = & 3.38065 \\ \log 65536 & = & 4.81648 \end{array}$$

$$\text{Difference} = 0.56417 - 2 = \log 0.03666$$

$$\begin{array}{rcl} \log 0.5133 & = & 0.71037 - 1 \\ \log 15015 & = & 4.17653 \end{array}$$

$$\begin{array}{rcl} \text{Sum} & = & 3.88690 \\ \log 16384 & = & 4.21442 \end{array}$$

$$\text{Difference} = 0.67248 - 1 = \log 0.47041$$

$$\begin{array}{rcl} \log 0.8700 & = & 0.93952 - 1 \\ \log 1001 & = & 3.00043 \end{array}$$

$$\begin{array}{rcl} \text{Sum} & = & 2.93995 \\ \log 32768 & = & 4.51545 \end{array}$$

$$\text{Difference} = 0.42450 - 2 = \log 0.02658.$$

This gives for the sum of the positive terms

$$\begin{array}{r}
 0.00011 \\
 0.03666 \\
 0.47041 \\
 0.02658 \\
 \hline
 S_2 = 0.53376
 \end{array}$$

The final result of the interpolation is found by subtracting the sum of the negative terms from the sum of the positive terms, which gives

$$\begin{array}{r}
 S_2 = 0.53376 \\
 S_1 = 0.09772 \\
 \hline
 0.43604.
 \end{array}$$

We find, therefore, that the value of the psychometric function of the "heavier"-judgments for 95 is for this subject 0.43604 or rather 0.4360 since the last figure is not exact. This is the value given in Table 90 (Appendix p. 220); all the values given in the tables were found by a similar computation. We see that the value of the psychometric function for 95 is smaller than $\frac{1}{2}$ and for 96 it is greater than $\frac{1}{2}$. The value of x for which the psychometric function assumes the value $\frac{1}{2}$ lies therefore in this interval. The difference between the values of the function for 95 and 96 is 0.0773 and the difference of 0.4360 from 0.5000 is 0.0640, and since $0.0640:0.0773=0.83$ we have for the value in question the approximate result 95.83. The same computation may be used for finding the just perceptible negative difference. These results are given for all the subjects in the Tables 93 and 94 (Appendix p. 221). The first two columns give the observed results of the threshold, and the values calculated by the algorithm of the method of just perceptible differences. The last column is marked "Value found by interpolation" and contains the results calculated in the way just described. The results of the observations and the values calculated by the different methods are sensibly equal in all the cases. The differences between the observed results and the values found by interpolation are, gen-

erally speaking, greater than those between the observed results and the values found by the algorithm of the method of just perceptible differences. The value found by interpolation coincides only in the case of subject IV so closely with the observed result that the difference is considerably smaller than that of the result calculated by the algorithm of the method of just perceptible differences, although the difference between the latter result and the observed result is only 0.16. This fact that the result of the method of just perceptible differences coincides better with the observations is not surprising, because in our experiments only one set of differences was used so that the chance influence of the choice of the set of comparison stimuli is very great. To this fact it is due that the method of just perceptible differences gives not the exact values for which $p = \frac{1}{2}$ or $u = \frac{1}{2}$. The results of the interpolation allow us to determine the interval of uncertainty in a new way; we give these quantities alongside with those found by the algorithm of the method of just perceptible differences.

Subject.	Interval of uncertainty by method of just perceptible differences.	Interval of uncertainty by interpolation.
I	6.15	7.69
II	3.96	4.33
III	1.43	1.57
IV	2.69	3.02
V	2.67	2.08
VI	4.56	5.22
VII	4.07	5.44.

The interpolated values deserve more credit than the results of the method of just perceptible differences, if the latter is not based on the results of experiments with different sets of comparison stimuli. The values found by interpolation are greater than the results of the method of just perceptible differences for all the subjects except for subject V. It is not void of interest to order the subjects by the size of the interval of uncertainty and to compare this order with the one which is obtained if the sub-

jects are ordered by the size of their probable errors which were calculated in the last chapter.

Subjects arranged by
probable error.

I
VII
II
VI
IV
V
III

Subjects arranged by
interval of uncertainty found by
interpolation.

I
VII
VI
II
IV
V
III.

The order of all the subjects is the same in both series except for the subjects II and VI who change places. The probable errors for those two subjects are only slightly different and one may account for this break in the regularity of the two series by chance disturbances of the empirical determination. If this explanation is accepted one comes to the conclusion that the probable error derived from the "equality"-judgments by the method described in the preceding chapter gives a measurement of the sensations which goes parallel to that afforded by the threshold.

We turn now to the study of the psychometric function for the "equality"-judgments. The curves representing these functions comply with our expectation in so far as their general trend is to rise at first and then to decrease after having attained a maximum. They have in common with the curves representing the two other psychometric functions that their course is irregular in the extreme parts of the charts. The fact that the curves do not rise to any great height is only an expression of the observation which was made previously that the relative frequency of the equality cases is very low even in that interval where it is greatest. It is interesting to make the following remark. The points E and E' in the charts mark the points of intersection with the curves for the psychometric functions of the "lighter" and of the "heavier"-judgments, the letter E being given to the point of intersection on the left. The curve for the "equality"-

cases rises above the curves for the 'heavier' and for the "lighter"-judgments in those cases where the point of intersection with the curve for the "lighter"-judgments lies on the left; this statement is, of course, restricted to points of intersection which lie within the interval of regularity. The point of intersection of the psychometric function for the "equality"-judgments with the curve for the "lighter"-judgments lies on the left of the point of intersection with the curve for the 'heavier'-judgments only in the cases of the subjects I, VI, and VII. There exists for these subjects an interval of subjective equality, which in the charts is marked by the letters E and E', within which there exists a greater probability for an "equality"-judgment than for any other kind of judgment. One must make the distinction between an interval of uncertainty and an interval of subjective equality. All subjects have an interval of uncertainty, but only those subjects have an interval of subjective equality for whom there exists an interval inside of which the probability of an "equality"-judgment exceeds the probabilities for "heavier" or "lighter"-judgments.

The curves for the "equality"-judgments approach more or less the type of a bell shaped curve inside an interval which is rather extended. The curve for the "equality"-judgments of subject IV resembles as much a probability curve as one may expect from an empirical determination; there occurs only one break at the extreme upper end of the curve. The curve for subject II comes almost as near to the type of a probability curve. With the exception of subject I all the curves approach the type of a probability curve within a considerable interval and even for this subject the course of this curve comes very near to that of a probability curve within the interval from 98 to 104. Outside the interval of regularity one obtains as a rule values which are too high, a fact which may be explained by the remark made above, that according to the observations of previous investigators the psychophysical conditions for the judgments on the comparison of two weights are not the same in series which contain only small differences as they are in series which contain large and small differences. Variations in the adaptation of attention are possibly the cause of this fact.

It is of some importance and not void of interest to find the maximum value of the psychometric function of the "equality"-judgments. The situation of this maximum throws some light on the question of what we have to understand by the point of subjective equality, and the maximum value of the psychometric function of the "equality"-judgments can be put to a very interesting use. The exact determinations of the value of x for which the psychometric function attains its maximum requires that the function be actually set up by Lagrange's equation. The maximum is found by differentiating this equation and solving the resulting equation of fifth degree. This process is very laborious and it seems to be advisable to use the following method. One finds in the tables of the psychometric function of the equality judgments the three values of x for which the function is greatest. Let us call the value which precedes the maximum A , the maximum value B and the third value C , and let the corresponding values of x be x_1 , x_2 and x_3 . These values are found approximately in the middle of the tables and they are always in the interval of regularity, so that all the values preceding A are smaller than A and all the values following B are smaller than B . We lay through these three points a parabola

$$f(x) = ax^2 + bx + c$$

in such a way that x_2 is the origin of the coordinates. This gives

$$f(-1) = A$$

$$f(0) = B$$

$$f(1) = C$$

which is a system of three linear equations for the determination of the three coefficients a , b and c . The solution gives

$$a = \frac{A + C}{2} - B$$

$$b = \frac{C - A}{2}$$

$$c = B.$$

The maximum of the function $f(x)$ is found by

$$f'(x_m) = 2ax_m + b = 0$$

which gives for x_m

$$x_m = -\frac{b}{2a}$$

or if we express the values of a and b by A , B and C

$$x_m = \frac{A-C}{2(A+C-2B)}$$

The maximum of the psychometric function is then determined by $x_2 + x_m$. This value lies on the side of x_1 or x_3 according to whether x_m has the negative or the positive sign. (1.) The sign of x_m depends on the sign of the difference $A-C$ if B is the greatest value, and the maximum value obviously is situated nearer to the value for which the function assumes the greater value. (2.) If A is the greatest value, the difference $A-C$ is positive and the sign of x_m depends on the difference $A+C-2B$. This difference is necessarily negative for a concave function decreasing, throughout the interval, i. e. for a function which decreases more rapidly than a straight line, as it is the case of the psychometric function of the "equality"-judgments in the interval of regularity for values of x which are greater than the value for which the maximum is attained. x_m is negative and the maximum of the psychometric function lies on the side of x_1 . (3.) If C is the greatest of our three values the difference $A-C$ has the minus sign and $A+C-2B$ must be negative, because the psychometric function of the "equality"-judgments is an increasing concave function inside the interval of regularity for values of x which are smaller than the value for which the maximum is attained. x_m is positive in this case and the maximum of the psychometric function lies on the side of x_3 .

The course of this simple calculation may be illustrated by working out the results for subject I. We find in Table 86 that the psychometric function of the "equality"-judgments for this subject has the greatest values for 97, 98 and 99. We therefore have to put

$A=0.4664$ which is the value of the function for $x=97$,
 $B=0.4827$ " " " " " " $x=98$,
 $C=0.4846$ " " " " " " $x=99$.

These quantities serve for the determination of x_m . We find from these data

$$\begin{aligned} A-C &= 0.0182 \\ 2(A+C-2B) &= 0.0288 \\ x_m &= +0.63. \end{aligned}$$

The maximum of the psychometric function is, therefore, situated at 98.63.

It is of some importance to get an idea about the degree of approximation of this calculation. It is, of course, possible to determine by purely theoretical considerations the error which is due to the approximate representation of the course of a function of sixth degree by a parabola, but it is perhaps best to show by an example what this error may be. This example will show that a considerable amount of work is required in order to obtain a determination not quite as good as ours, if the maximum is found by differentiating the equation set up by Lagrange's formula. This is due to the fact that the coefficients of the equation in our example were not determined accurately enough, because if they had been, the determination of the maximum would be absolutely exact. This example serves also the ulterior purpose of showing that the application of Lagrange's formula is a little cumbersome for some purposes although, or rather because, this formula represents the smallest possible theoretical addition. The idea suggests itself that a more definite assumption might lead to simpler results. It seems convenient to postpone the discussion of this question for a future publication, although we might be in the possession of the solution of this problem.

We will determine the position of the maximum of the psychometric function of the "equality"-judgments for the subject I by setting up this equation by the formula of Lagrange. The expression of Lagrange's formula has the form

$$\begin{aligned}
 F(x) = & \frac{(x-88)(x-92)(x-96)(x-100)(x-104)(x-108)}{4 \cdot 8 \cdot 12 \cdot 16 \cdot 20 \cdot 24} \cdot 0.0622 - \\
 & - \frac{(x-84)(x-92)(x-96)(x-100)(x-104)(x-108)}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 16 \cdot 20} \cdot 0.1244 + \\
 & + \frac{(x-84)(x-88)(x-96)(x-100)(x-104)(x-108)}{8 \cdot 4 \cdot 4 \cdot 8 \cdot 12 \cdot 16} \cdot 0.3311 - \\
 & - \frac{(x-84)(x-88)(x-92)(x-100)(x-104)(x-108)}{12 \cdot 8 \cdot 4 \cdot 4 \cdot 8 \cdot 12} \cdot 0.4422 +
 \end{aligned}$$

$$\begin{aligned}
& + \frac{(x-84)(x-88)(x-92)(x-96)(x-104)(x-108)}{16.12.8.4.4.8} .04644- \\
& - \frac{(x-84)(x-88)(x-92)(x-96)(x-100)(x-108)}{20.16.12.8.4.4} .00911+ \\
& + \frac{(x-84)(x-88)(x-92)(x-96)(x-100)(x-104)}{24.20.16.12.8.4} .00533.
\end{aligned}$$

It is of advantage for the actual setting up of the equation to write the formula in this way

$$\begin{aligned}
F(x) = & \frac{\phi(x)}{x-84} \cdot \frac{0.0622}{2^{16} \cdot 3^2 \cdot 5} - \frac{\phi(x)}{x-88} \cdot \frac{0.1244}{5^{15} \cdot 3 \cdot 5} + \frac{\phi(x)}{x-92} \cdot \frac{0.3311}{2^{16} \cdot 3} - \\
& - \frac{\phi(x)}{x-96} \cdot \frac{0.422}{2^{14} \cdot 3^2} + \frac{\phi(x)}{x-100} \cdot \frac{0.4644}{2^{16} \cdot 3} - \frac{\phi(x)}{x-104} \cdot \frac{0.911}{4^{15} \cdot 3 \cdot 5} + \\
& + \frac{\phi(x)}{x-108} \cdot \frac{0.0533}{2^{16} \cdot 3^2 \cdot 5}
\end{aligned}$$

where

$$\phi(x) = (x-84)(x-88)(x-92)(x-96)(x-100)(x-104)(x-108)$$

Carrying out all the multiplications indicated gives

$$\begin{aligned}
\phi(x) = & x^7 - 672x^6 + 193312x^5 - 30858240x^4 + 2952081664x^3 - \\
& - 169250070528x^2 + 5384524382208x - 73329396940800.
\end{aligned}$$

Dividing by the linear factors one obtains the expressions needed.

$$\begin{aligned}
\frac{\phi(x)}{x-84} = & x^6 - 588x^5 + 143920x^4 - 18768960x^3 + 1375489024x^2 - \\
& - 53708992512x + 872969011200
\end{aligned}$$

$$\begin{aligned}
\frac{\phi(x)}{x-88} = & x^6 - 584x^5 + 141920x^4 - 18369280x^3 + 1335585024x^2 - \\
& - 51718588416x + 833288601600
\end{aligned}$$

$$\begin{aligned}
\frac{\phi(x)}{x-92} = & x^6 - 580x^5 + 139952x^4 - 17982656x^3 + 1297677312x^2 - \\
& - 49863757824x + 797058662400
\end{aligned}$$

$$\frac{\phi(x)}{x-96} = x^6 - 576x^5 + 138016x^4 - 17608704x^3 + 1261646080x^2 -$$

$$49129046848x + 763847884800$$

x which is done conveniently by adding up the positive and the negative terms separately and forming the differences of these sums. This computation is given on the insert opposite page 129. The lines are marked $\frac{a_k \phi(x)}{m_k(x-x_k)}$ and contain the coefficients of the powers of **x** which stand at the head of each column. The terms

$\frac{a_{2k+1}\phi(x)}{m_{2k+1}(x-x_{2k+1})}$ are positive and the term $\frac{a_{2k}\phi(x)}{m_{2k}(x-x_{2k})}$ are negative and their coefficients must be added up separately.

x	
- 1132.778365833333333333333333	+ 18411.8220
83973.64408125000000000000	1342295.9550
109864.87852500000000000000	1732084.7544
787.44687125000000000000	12271.2590
195758.7478433333333333333333	3105063.7904
165770.33219875000000000000	2632256.2980
- 29988.4156445833333333333333	+ 472807.4924
- 29988.4156445833333333333333	
x	
- 13089.5841450000000000000000	+ 210899.0520
144341.30260000000000000000	2290673.2850
8339.44545375000000000000	+ 130683.8610
- 165770.33219875000000000000	+ 2632256.2980

98.571 is a possible maximum of the function. This result is by 0.06 smaller than that of our approximate determination. These two determinations of the maximum may be compared with the actual course of the function in the interval from 98.55 to 98.67. By effecting the interpolation in the usual way we find the following results

Value of x.	Value of the psychometric function of the "equality"- judgments for subject I.
98.55	0.48600
98.57	0.48602
98.59	0.48605
98.61	0.48605
98.63	0.48604
98.65	0.48603
98.67	0.48602.

This little table shows that neither 98.57 nor 98.63 is the maximum of the function. The maximum lies in the interval between 98.59 and 98.61 nearer to the second value than to the first; 98.63 is, therefore, a more exact determination of the maximum than 98.57. Such a difference, however, is insignificant for our purpose, because the maximum value of the psychometric function of the "equality"-judgments is of greater interest than the value of x for which the maximum is attained. The result of the approximate determination of the maximum certainly falls in the neighborhood of the real maximum, and the determination of the greatest value which the psychometric function attains is not considerably affected by a small error, because a function varies only little in the neighborhood of its maximum.

The results of the approximate determination of the position of the maximum of the psychometric function of the equality cases for our seven subjects are given here alongside with the arithmetical means of the "equality"-judgments which were found previously.

Subject	Maximum	Arithmetical Mean	Difference
I	98.61	96.10	+2.51
II	97.57	96.43	+1.14
III	100.51	97.78	+2.73
IV	97.34	97.15	+0.19
V	95.89	96.98	-1.09
IV	99.23	97.65	+1.58
VII	96.80	98.03	-1.23.

The maximum comes nearest to the arithmetical mean for the results of subject IV; this is only a consequence of the fact that the curve for this subject is very regular and approaches the symmetrical type. It is interesting to notice that the interval inside of which the maximum values vary for the seven subjects is by far greater than the interval within which the arithmetical means vary. The smallest value of the maximum is 95.89 and the largest is 100.51, which gives 4.62 for the length of the interval. The arithmetical means vary between 96.10 and 98.01, or in an interval of 1.91. The values of x for which the psychometric function for the "equality"-judgment attains its maximum are those values which have the highest probability of being judged equal to the standard stimulus. This value has to be taken as a determination of the point of subjective equality. It is of interest to notice that the point of subjective equality does not coincide with the arithmetical mean. The value for which the function reaches the maximum is in five cases greater and in two cases smaller than the arithmetical mean; the difference between these two values is 1.50 on the average. The intensities of the comparison stimulus for which the maximum values are reached are generally smaller than the standard weights of 100 gr; only subject III is an exception since for this subject the maximum is attained for 100.51. This indicates that the second weight is overestimated as a rule. The amount of overestimation, however, is smaller if we take the value for which the "equality"-judgments have the highest probability as representative of the point of subjective equality, than it is if we take the arithmetical mean.

The next step is to find the maximum value of the function. This value is found by introducing the value of x for which the function attains its maximum into Lagrange's formula. The values of x for which the psychometric function of the "equality" judgments reaches its maximum are different for the different subjects. For this reason the computation must be arranged in a way slightly different from the one described above and it is not possible to attack the results for all the subjects at once. The computation is considerably facilitated by the circumstance that the observations are made for all the subjects on the same comparison stimuli. It is a consequence of this fact that the con-

stant denominators of the terms in Lagrange's formula are the same for all the subjects. Let us call these denominators $m_1, m_2, \dots, m_7$, so that M_i is the constant denominator of the coefficient of a_i . In working out the results for subject I we begin by finding the logarithms of these quantities and obtain

$$\begin{aligned}\log m_1 &= 6.46969 \\ \log m_2 &= 5.69154 \\ \log m_3 &= 5.29360 \\ \log m_4 &= 5.16866 \\ \log m_5 &= 5.29360 \\ \log m_6 &= 5.69154 \\ \log m_7 &= 6.46969.\end{aligned}$$

We have found by our abbreviated method that the psychometric function of the equality judgments for the subject I reaches its maximum for $x' = 98.63$. We have to form the products

$$\begin{aligned} & (x' - 88)(x' - 92)(x' - 96)(x' - 100)(x' - 104)(x' - 108) \\ & (x' - 84)(x' - 92)(x' - 96)(x' - 100)(x' - 104)(x' - 108) \\ & (x' - 84)(x' - 88)(x' - 96)(x' - 100)(x' - 104)(x' - 108) \\ & (x' - 84)(x' - 88)(x' - 92)(x' - 100)(x' - 104)(x' - 108) \\ & (x' - 84)(x' - 88)(x' - 92)(x' - 96)(x' - 104)(x' - 108) \\ & (x' - 84)(x' - 88)(x' - 92)(x' - 96)(x' - 100)(x' - 108) \\ & (x' - 84)(x' - 88)(x' - 92)(x' - 96)(x' - 100)(x' - 104)\end{aligned}$$

The simplest way to find these products is to form the product

$$P = (x' - 84)(x' - 88)(x' - 92)(x' - 96)(x' - 100)(x' - 104)(x' - 108)$$

and to subtract the single factors. Introducing the value 98.63 for x' we obtain:

$$\begin{aligned}\log 14.63 &= 1.16524 \\ \log 10.63 &= 1.02653 \\ \log 6.63 &= 0.82151 \\ \log 2.63 &= 0.41996 \\ \log 1.37 &= 0.13672 \\ \log 5.37 &= 0.72997 \\ \log 9.63 &= 0.97174 \\ \hline \log P &= 5.27167\end{aligned}$$

The logarithmic computation takes the following form:

$$\begin{array}{l} \log a_1 = 0.79379 - 2 \\ \log P = 5.27167 \end{array}$$

$$\begin{array}{l} \Sigma_1 = 4.06546 \\ \Sigma_2 = 7.63493 \end{array}$$

$$\begin{array}{l} \log m_1 = 6.46969 \\ \log 14.63 = 1.16524 \end{array}$$

$$\Sigma_2 = 7.63493$$

$$\Sigma_1 - \Sigma_2 = 0.43053 - 4 = \log 0.00027$$

$$\begin{array}{l} \log a_2 = 0.09482 - 1 \\ \log P = 5.27167 \end{array}$$

$$\begin{array}{l} \Sigma_1 = 4.36649 \\ \Sigma_2 = 6.71807 \end{array}$$

$$\begin{array}{l} \log m_2 = 5.69154 \\ \log 10.63 = 1.02653 \end{array}$$

$$\Sigma_2 = 6.71807$$

$$\Sigma_1 - \Sigma_2 = 0.64842 - 3 = \log 0.00445$$

$$\begin{array}{l} \log a_3 = 0.51996 - 1 \\ \log P = 5.27167 \end{array}$$

$$\begin{array}{l} \Sigma_1 = 4.79163 \\ \Sigma_2 = 6.11511 \end{array}$$

$$\begin{array}{l} \log m_3 = 5.29360 \\ \log 6.63 = 0.82151 \end{array}$$

$$\Sigma_2 = 6.11511$$

$$\Sigma_1 - \Sigma_2 = 0.67652 - 2 = \log 0.04748$$

$$\begin{array}{l} \log a_4 = 0.64562 - 1 \\ \log P = 5.27167 \end{array}$$

$$\begin{array}{l} \Sigma_1 = 4.91729 \\ \Sigma_2 = 5.58862 \end{array}$$

$$\begin{array}{l} \log m_4 = 5.16866 \\ \log 2.63 = 0.41996 \end{array}$$

$$\Sigma_2 = 5.58862$$

$$\Sigma_1 - \Sigma_2 = 0.32867 - 1 = \log 0.21314$$

$$\begin{array}{l} \log a_5 = 0.66689 - 1 \\ \log P = 5.27167 \end{array}$$

$$\begin{array}{l} \Sigma_1 = 4.93856 \\ \Sigma_2 = 5.43032 \end{array}$$

$$\begin{array}{l} \log m_5 = 5.29360 \\ \log 1.37 = 0.13672 \end{array}$$

$$\Sigma_2 = 5.43032$$

$$\Sigma_1 - \Sigma_2 = 0.50824 - 1 = \log 0.32229$$

$$\log a_6 = 0.95952-2$$

$$\log P = 5.27167$$

$$\Sigma_1 = 4.23119$$

$$\Sigma_2 = 6.42151$$

$$\Sigma_1 - \Sigma_2 = 0.80968-3 = \log 0.00645$$

$$\log a_7 = 0.72673-2$$

$$\log P = 5.27167$$

$$\Sigma_1 = 3.99840$$

$$\Sigma_2 = 7.44143$$

$$\Sigma_1 - \Sigma_2 = 0.55697-4 = \log 0.00036$$

$$\log m_6 = 5.69154$$

$$\log 5.37 = 0.72997$$

$$\Sigma_2 = 6.42151$$

$$\log m_7 = 6.46969$$

$$\log 9.68 = 0.97174$$

$$\Sigma_2 = 7.44143$$

The terms resulting from a_1, a_3 and a_6 have to be taken with the negative sign; the sum of these terms is

$$0.00027$$

$$0.04748$$

$$0.00645$$

$$S_2 = 0.05420.$$

The sum of the terms a_2, a_4, a_5 , and a_7 , which have the positive sign, is

$$0.00445$$

$$0.21314$$

$$0.32229$$

$$0.00036$$

$$S_1 = 0.54024$$

$$S_2 = 0.05420$$

$$S_1 - S_2 = 0.48604.$$

The difference $S_1 - S_2 = 0.4860$ is the value which the psychometric function of the "equality"-judgments for subject I attains at the point $x = 98.63$.

We give here the results for all our subjects indicating the value for which the maximum is reached and the value which the psychometric function attains at this point.

Subject	Position of the maximum	Maximum value of the psychometric function
I	98.63	0.4860
II	97.57	0.2667
III	100.51	0.1508
IV	97.34	0.2111
V	95.89	0.1969
VI	99.23	0.3771
VII	96.80	0.3797.

This table shows clearly that the values of the psychometric function of the "equality"-judgments do not exceed $\frac{1}{2}$ and, except for subject I, they even do not come anywhere near this value. Let us arrange our seven subjects by the maximum value which their psychometric function for the "equality"-judgments attains, and let us compare this order with those which we obtain when we arrange the subjects (1) by the length of the interval of uncertainty determined by the method of interpolation, (2) by the length of the interval of uncertainty determined by the method of just perceptible differences, and (3) by the probable error as determined in the preceding chapter.

Order of subjects by maximum of the psychometric function of the "G"-judgments.	Order of subjects by interval of uncertainty de- termined by interpolation.	Order of subjects by interval of un- certainty deter- mined by method of just percepti- ble differences.	Order of sub- jects by prob- able error.
I	I	I	I
VII	VII	VI	VII
VI	VI	VII	II
II	II	II	VI
IV	IV	IV	IV
V	V	V	V
III	III	III	III.

This table shows that the order of the subject is the same in the first and second column; every subject appears at the same place in these two columns. We conclude from this fact that the maximum value of the psychometric function may serve as a measure of the sensitivity of a subject just as well as the interval of uncertainty as determined by interpolation. The order of the subjects by the length of the interval of uncertainty as determined by the method of just perceptible differences and the order by the probable error are slightly different from those preceding. All the four orders, however, become absolutely identical if subject VI is omitted. This subject is the only one who is out of his place. It is a matter of course that the determinations of the interval of uncertainty by the method of just perceptible differences and by interpolation must give similar results, because both methods determine the same quantity and their results must coincide within the limits of the accuracy of an empirical determination. It is more surprising that these results coincide with those obtained by the determinations of the probable error and of the maximum value of the psychometric function of the "equality"-judgments. These two quantities depend entirely on the "equality"-judgments, whereas the threshold in the direction of increase depends on the probability of "greater"-judgments and the threshold in the direction of decrease depends on the probability of "smaller"-judgments. The threshold, therefore, does not depend directly on the frequency of the "equality"-judgments. The probable error and the maximum of the psychometric function of the "equality"-judgments, furthermore, are found by algorithms which have nothing in common. We conclude from the fact, that the results of these four different methods coincide, that there must exist some kind of a relation between the values of the different psychometric functions, and there must also exist a relation between the probable error and the maximum value of the psychometric function of the "equality"-judgments. From these facts one can draw certain conclusions as to the nature of the three psychometric functions. The fact that the length of the interval of uncertainty, the probable error and the maximum of the psychometric function of the "equality"-judgments are quantities each one of which may serve as a measure of the sensitivity of a

subject is an important point for the special study of the psychometric functions. One would be free to make an appropriate hypothesis for the explanation of this fact, if one chooses to do so. It is, however, doubtful whether there is any great use in introducing a hypothesis at this point. One reason for introducing a hypothesis consists in simplifying the practical application of a method. The determination of the limits of the interval of uncertainty by interpolation and the computation of the probable error are so handy, that it is not likely that any hypothesis could simplify them very much or define a measure of sensation which is more readily to be found. Another reason for introducing a hypothesis is the interest which we take in connecting different empirical facts, because we gain a new insight by establishing relations between facts. There exists, however, no such need at present and it seems, therefore, to be best to leave this fact for further analysis without mixing it up with any hypothesis. Such a hypothesis would refer very likely to the nature of the dependence of the probabilities of the different classes of judgments on the intensity of the comparison stimulus. A definite assumption on the nature of the psychometric function of the "equality"-cases is indispensable for those sciences which use the results of repeated observations on the same quantity. Psychology has at present no practical interest in making such an assumption. There remains the theoretical interest in the problem, which can be satisfied only by the results of further analysis and not by a hypothesis. The special study of the psychometric functions will be the object of a future publication. We may sum up the peculiarly advantageous position of psychology by saying that the dependence of the probabilities of the different classes of judgments on the intensity of the comparison stimulus is an object of investigation for psychology, and that it need not be made the object of a hypothesis. This problem can be treated only accidentally in other sciences, because all that is needed there is a fixed standard by which the accuracy of different determinations can be compared, and a definite algorithm for the combination of observations which do not give identical values for the same quantity.

We have seen that the probable error and the maximum probability for an "equality"-judgment may be used for a measure-

ment of the accuracy of sensation in the same way as the interval of uncertainty. These quantities are entirely based on the probabilities of the "equality"-cases. This fact is, perhaps, a little surprising if one remembers that this class of judgments was more or less of a difficulty for the psychophysical methods, a difficulty which seemed to be so serious that some investigators recommended suppressing these judgments entirely.

CHAPTER VI.

A GENERAL INQUIRY CONCERNING THE PSYCHOMETRIC FUNCTIONS.

We have avoided introducing into the above discussion any one of those notions which in most of the current text-books of psychology are proclaimed as indispensable presuppositions of this science. There is, for this reason, no necessity for philosophical considerations in our argumentation. The notions of which we have made use are by no means different from those used e. g. in the theory of life insurance or in the formal theory of population. There is, however, one point in our considerations which seems to throw considerable light on the epistemological nature of some scientific methods and which for this reason deserves some attention.

The point in question is the introduction of the assumption that the probabilities of the different judgments are analytic functions of the amount of difference of the intensities of the stimuli. This assumption leads to the construction of the notion of a psychometric function which was found useful for many purposes. We confined our discussions to the considerations of the case that an algebraic function of sixth degree expresses the results of our observations on seven values, because this assumption is the smallest possible theoretical addition. The degree of this equation depends merely on the number of the values for which observations were made and the mathematical expression may be adapted to any number of observations whatsoever. Interpolating by Lagrange's formula has not the character of a definite hypothesis on the nature of the psychometric function, but it is rather a means of completing a set of observations. The method of expressing a set of n observations by an algebraic function of degree $n-1$ lends itself more readily to different results than any other hypothesis; it is of course indifferent whether the values of these functions are determined by Lagrange's formula of inter-

polation or whether Newton's method of differences is used. We have seen, on the other hand, that an algebraic function can be only a preliminary result, because it cannot possibly give the dependence of the probabilities of the judgments of different types for all the values of the comparison stimulus, since an algebraic function must assume values greater than the unit or smaller than zero. The question as to the nature of this dependence of the probabilities on the intensity of the comparison stimulus can be settled definitively only by experimental evidence; as long as no facts are at hand one is free to introduce any hypothesis which may seem appropriate. The following considerations will show that some caution is recommended, because the assumption of a definite form of the psychometric function is a far reaching hypothesis.

The immediate result of the observations is a set of results for a greater or smaller number of observed differences. The number of observations must be of course sufficient to determine the constants of the psychometric function the form of which is supposed to be known; if the number of observations is greater than the number of parameters in the psychometric functions one may combine the results and determine the most probable value of the constants, but the problem remains indeterminate if the number of observations is smaller than the number of constants of the function. After the constants of the function are found one can give the probabilities for all the intensities of the comparison stimulus. This means that a finite number of observations—in many cases a very small number—is sufficient to serve as a basis for an infinity of statements. The probabilities of the different classes of judgments depend on the psychophysical condition of the subject under observation, and in the absence of more complete information we can characterize the psychophysical condition of a subject only by the probabilities with which the different classes of judgments may be expected for various differences. It would seem that an infinite number of observations were needed for this purpose, because there is no reason a priori why the group of causes which gives a certain probability to the judgment "greater" on the difference x should be in any connection with the group of causes which determine this probability

for the difference y . The psychophysical conditions of a subject which determine the probabilities of the judgments of different classes on a given difference come obviously under the heading of what is usually called an attribute or a quality. The existence of a psychometric function which can be determined by a finite number of observations implies that there exist certain relations between the different qualities, so that all the qualities can be expressed by a certain number of them. It is obvious that no experience whatsoever can be the basis of such a statement, and it follows from this that the notion of a psychometric function is not a result of experience, but the expression of the methodological assumption that there exist relations of some kind between the different qualities of the subject. There is no doubt that the conclusions drawn from this assumption are in agreement with experience and one therefore can only ask: Why is it that one can conclude from one quality of an object to another quality? Why must an object have certain qualities because it has certain other qualities? The formal character of the logical process is that of a subsumption. The general expression of the psychometric function is the major which is specified by introducing special values. The difficulty is not of logical but of epistemological nature and refers to the way in which the psychometric function is established. It is quite obvious that we never can reach a proposition of such generality by induction, because the psychometric function makes a statement about every element of a class which is of the order of the continuum, and, as a matter of fact, the number of observations on which this statement is based is rather small. It is, therefore, not a generality of the empirical type. It is, furthermore, clear that the relations which exist between the different qualities of a subject are not of the causal type, if this word is used in its common meaning which refers to a succession of phenomena. Qualities of an object are simultaneous, not successive, and the relation of qualities, therefore, cannot possibly be of the causal type. It does not avail, either, to take the view that the qualities which determine the probabilities of a judgment of a certain type for different intensities of the comparison stimulus must follow a certain law, because they are qualities of the same subject. The only rigorous condition that

must be satisfied by qualities of the same object is that they are not contradictory. There is nothing in the qualities which determine the probability of a "heavier"-judgment for the difference x which implies, that there must be a certain other probability for another difference y . Such a conclusion can only be made after the dependence between these qualities is established by means of the psychometric function, and one will try in vain to formulate the problem in such a way, that the probability of a "heavier"-judgment for the difference y follows from that for the difference x by logical processes, if one does not use the notion of a psychometric function.

This problem, which is decidedly of philosophic nature, is little treated in the literature and it does not seem that anybody has given a solution besides that of Kant. This solution may be formulated in this way. The empirical datum are certain observed values which, in themselves, are not a law nor are they in relation to each other. Such a relation can be established merely by representing these data in a possible intuition.* The empirical representative of this intuition, in our case, is supplied by representing the results by points in a plain and by a curve drawn through them. The representation in a pure intuition makes it possible to find a law of the generality of the psychometric functions. Without trying to refute or to confirm Kant's solution we will try to show by the following considerations that the question of the relation between the qualities of an object implies a problem of very great generality, which contains the epistemological problem of the principle of causality as a special case.

The totality of those facts which constitute experience are the material which science works on. There is no primary distinction between these facts, all of which stand on the same level of reality, but a great number of secondary distinctions comes in, which are introduced for the sake of the description of the phenomena, because the task of giving an exhaustive de-

*The word intuition has become customary as a translation of the German term "*Anschauung*," but this translation is not quite appropriate in so far as this word carries the side meaning of "as if by inspiration". For this reason it would be advisable to invent a new word for the translation of this purely technical term.

scription is made easier by classifying the events into groups. The term "science" is frequently used in two slightly different significations. In the first signification this term designates a group of propositions which describe a certain part of the realm of experience, but in its second meaning this term is applied to every effort or activity which has the purpose of establishing such a system. We shall speak of science only in the first signification as applied to a set of propositions which describe a given part of experience. We may speak of a system of science, if its single propositions are connected in such a way that all the propositions may be deduced from a set of propositions. We may speak in this case of a systematic science in contra-distinction to descriptive sciences which may be able to give an accurate description of all the phenomena in their realm of experience, but which are not able to deduce all their propositions from a fundamental set of propositions. A system of science is economical if the logical processes involved are reduced to the smallest possible number. The quality of being economical involves that the set of fundamental propositions is reduced to the smallest possible number; i. e. to those propositions which are necessary and sufficient to deduce all the propositions of the system. Propositions of this type are independent from each other; this quality can be proved by showing that not all the propositions of the system can be deduced, if one of the fundamental propositions, or a group of them, is omitted. Every proposition of a set of independent propositions may be substituted by its logical opposite without the conclusions drawn from this new set by purely logical processes being contradictory among themselves. The logical truth of a system of science is the consistence of any one of its propositions, or of any group of propositions, with any other proposition of the same system, or of any proposition which may be derived by purely logical processes from the same set of fundamental propositions. A set of fundamental propositions is logically neither true nor false* if its elements are independent of each other.

*For this reason they are sometimes called propositional functions, see B. RUSSELL, *Principles of Mathematics*, 13, 353; A. N. WHITEHEAD, *Introduction logique a la géométrie*, *Revue de Métaphysique et de Morale*, Vol. 15, 1907, p. 35.

The greatest and most perfect example of such a system of science is geometry, which has in so far a distinguished position among the other sciences as it can state at the start the propositions which will exhaust its field of work. It is due to the investigations into the foundations of geometry that one has acquired an insight into the relations of the single propositions to the original suppositions of a science, since it was seen that systems of consistent propositions may be derived from fundamental sets of propositions in which one proposition is superseded by its opposite. Physical sciences have at present not yet reached this high state of development, but there is little doubt that similar sets of necessary and sufficient propositions may be constructed for the different branches of mathematical physics, as it was required by Hilbert.* It is more doubtful whether it will be possible to construct similar sets for the biological sciences, because one has not yet found principles of high generality, and it is e. g. by no means an easy task to state all the principles which are common to physical and chemical physiology. The fact remains that one tries to construct systems which cover a certain range of the field of work.

These systems have nothing to do with the existence or non-existence of the objects they speak of. A system of science becomes applicable to a certain part of experience only by means of the statement, that the objects contained therein are such as those which are spoken of in this science. This application to experience is the test of the empirical truth of a system of science. A system of science is empirically true if an empirical fact corresponds to every one of its propositions, and the system is complete if there corresponds to every empirical fact of the realm of experience in question a certain proposition of the system. As a rule one concludes from the fact that certain empirical events comply with the requirements of the system that all its conclusions must be true, but one also can argue from the agreement of the conse-

**D. HILBERT, *Göttinger Nachr. Math.-phys. Kl.* 1900, p. 272 (also *Arch. f. Math. u. Phys.* Vol. I, (3 ser.) 1901, p. 62); O. HOELDER (*Anschauung und Denken in der Geometrie*, 1900) suspects that the deductions of mechanics might possibly be not as pure as those of geometry.

quences with experience to the truth of the system.* This logical process is under the general principle that all the conclusions drawn by merely logical processes from a proposition which is empirically true are empirically true. This may be illustrated by the example of geometry. Those objects which we call spatial manifoldnesses belong to three different types. The sets of fundamental propositions required for the description of these different manifoldnesses are identical in all but one proposition and, since this proposition is independent from the others, one may derive different systems of geometry, in which every proposition is consistent with any other proposition, or group of propositions, of the same system. All these systems, therefore, are logically true, but only one of them has empirical truth too. These three manifoldnesses differ in regard to a certain quantity, the so-called Gaussian measure of curvature, which is capable of empirical determination and the question which of the three systems of geometry is empirically true is reduced to the *quaestio facti*: Which is the value of the measure of curvature of empirical space? Those measurements which show that physical space has a curvature which can differ from zero only by a quantity which is smaller than the smallest which can be detected by our instruments of measurement, is the empirical warrant that all the conclusions of Euclidean geometry are empirically true. After a system of science is established and its empirical basis is put into proper light, it is possible to state it entirely as a series of consequences of a simple empirical fact, but our subjective confidence in such systems depends materially on the agreement in which its propositions are found to be with experience. The explanation of a

*L. BOLTZMANN, *Über die Entwicklung der Methoden der theoretischen Physik in neuerer Zeit, Jahresbericht d. deutschen Mathematiker-Vereinigung*, Vol. 8, 1899, p. 89, says that our confidence in the fundamental equations of electrodynamics is not so much based on Ampere's experiments, which allow us to state them as on the general agreement of experiment with all the conclusions which may be drawn from them. The case of geometry as science of space is very similar. Nobody doubts geometry but not because one depends so much on those facts which show that actual space, if it can be regarded as a geometrical manifoldness, is a Euclidean manifoldness, but because all the propositions of geometry have been found to correspond to geometrical facts relating to empirical space

phenomenon which belongs to such a system of science consists in the reduction of the proposition which is the description of this phenomenon to the system of fundamental propositions i. e. in showing how this proposition may be obtained as a logical conclusion from them.*

The calculus of probabilities, similarly, is a system of proposition derived by purely logical processes from the notion of a mathematical probability, which has its logical foundation in the hypothetical judgment. There exists no connection between this system and actual events and one tries in vain to deduce any such relation by means of the theorem of Bernoulli or of the theorem of Poisson. These propositions become applicable to a certain realm of experience only by means of the statement: The events of this realm of experience have the character of randomness in the mathematical sense of the word. It follows from this statement that all the propositions of the calculus of probabilities must be true for this group of events, but the truth of every single proposition is based and derived from this one empirical fact that the events have random character. The more frequent practice of concluding from the agreement of all the inferences of the calculus of probabilities with experience to the randomness of the events stands of course on the same epistemological level.

The set of fundamental proposition may be subjected to criticism or revision, but as long as one stands on the ground of this science one has no possibility, and no need, to doubt the truth of the set of fundamental propositions. All one can do is to state the fact that the conclusions drawn from it coincide or do not coincide with experience. It is not always possible to state the fundamental principles in such a way that one can be sure of their empirical truth; this is the case mostly when the facts science has to work on are yet imperfectly known. In these cases one accepts the formulation which fits best to the facts known; propositions

*C. F. GAUSS, *Magnetismus und Magnetometer*, Werke, Vol. 5, p. 315. "Unter Erklären versteht aber der Naturforscher nichts anderes, als das Zurückführen auf möglichst wenige und möglichst einfache Grundgesetze, über die er nicht weiter hinaus kann, sondern sie schlechthin fordern muss, und aus ihnen die Erscheinungen vollständig ableitet."

of this kind are hypotheses of generalizing character. Systems which are based on generalizations from incomplete data are likely to undergo changes due to better information. The corrections which our views have to undergo under the influence of the knowledge of new facts must render the system again the one that fits the facts best. This requirement is based on the methodological principle that a proposition based on the observation of empirical facts holds good for experience in general as long as no contrary instances are known. The system of science which fits best the facts known has, therefore, the smallest probability of being obliged to undergo a correction under the influence of new information. This rule may be formulated, that one chooses among systems of hypothetical character the one which has the greatest stability.

It was mentioned above that the difficulties which one encounters in the construction of these systems of science are very different in the different realms of experience. The totality of all the data which constitute experience may be divided into two classes. The first class comprises all those events which are interpreted as indicating a mental life similar to our own; the second class is constituted by all the events which are not interpreted in this way. Let us designate events of the first class by the capital letters A, B, C, ..., events of the second class by the small letters a, b, c, ... and let us use greek letters if we speak of an event without laying stress on its belonging to one of these two classes. The description of the phenomena a, b, c, ... has made steady progress, but that of the phenomena of the class A, B, C, ... was found to be singularly difficult. The elements of the class a, b, c, ... show a certain constancy in their arrangement which makes classification comparatively easy, and which is very important for the study of the succession of these phenomena, understanding by succession the temporal order of the phenomena. This succession is under the general rule of the law of causality which states that there are certain definite rules which regulate the order in which a group of events $\mu, \nu, \sigma, \dots$ is followed by other events $\mu', \nu', \sigma', \dots$. If the rules are such that a certain group $\rho, \sigma, \tau, \dots$ is always followed by the same group $\rho', \sigma', \tau', \dots$, we may say that the succession shows uniformity, but if the

rules are such that a group of events ρ , σ , τ , ... may be followed at different times by different groups, although the group which follows it, is well defined, we may say that the succession is regular. The law of causality in this general formulation does not discriminate between events of the classes A, B, C and a, b, c, ... The difficulty of establishing rules of succession depends to a large extent on the possibility of identifying groups of events and on their stability, which consists chiefly in their not being broken up or interrupted by other events. The number of rules for the succession of the events of the class a, b, c, ... is so great and that for the class A, B, C, ... is so small, that one has tried to utilize this fact for laying down a general distinction between these two classes of events. Leaving the metaphysical side of this view out of question (that the class A, B, C, ... is made up of those events for which it is characteristic that they never can be brought under the law of causality) this definition is certainly not serviceable for empirical purposes, because the apparent lack of rules of the succession is also found in other events (events of random character) which we do not classify among A, B, C, ... Drawing the line of demarcation at those points where the limits of our actual knowledge are, has the inconvenience that the class A, B, C, ... is cut down by every progress of science. The logical outcome of such a procedure is the denial of the existence of this class, the elements of which are found only outside of the realm of science, a denial, which in turn is confronted with the fact that these events, whatever their general importance may be, are most intimately known and of greatest immediate interest for us.

The difference between these two classes does not exist for primitive thinking, which is anthropomorphic to such an extent that all the events seem to belong to the class A, B, C, ... The above mentioned elimination of this class belongs only to a comparatively late stage of development. This development is brought about by the consistent and continuous use of certain notions the origin of which we do not discuss here and the metaphysical bearing of which we leave aside. In the first rank among these notions stand the ideas of substance and causality. Substance is defined as that, the idea of which is the absolute subject of our

judgments and which is, therefore, not a determination of anything. Substance is not the predicate of any object, but it is the subject to which all attributes refer. All phenomena are regarded as attributes of something, and this is substance which remains unaltered by the change of its determinations. The difference between the two classes of events being established, the phenomena were referred to two entirely different substances, and the problem was to define them and to answer the questions which arise from the application of these notions to experience. The bearer of all the predicates of the class $a, b, c, \dots$ can be defined as what is moveable in space, and the term used for this notion is matter. The substance underlying the phenomena $A, B, C, \dots$ is characterized by thinking. These are the characteristic qualities of the *materia extensa* and of the *materia cogitans*, from which their other determinations follow.

These notions of mind and matter met with very different success. On the notion of matter one can build up systems of almost inexhaustible fertility, but all the efforts to deduce anything from the notion of a thinking substance have been in vain. In the realm of experience this notion is of no use, and beyond this it leads only to endless controversies. The success of the one notion, however, produced a favorable prejudice in favor of the other and numberless attempts were made to define it in such a way that the difficulties which it implies may be avoided.

We may call every theory which uses the notion of substance an ontological hypothesis, because it expresses a view on the nature of things. Any ontological theory must take one of the following views. 1.) There exists only one substance, because the other can be reduced to it; 2.) there exist two substances, one of which produces the phenomena $A, B, C, \dots$ and the other the phenomena $a, b, c, \dots$. The first hypothesis is the monistic view, which may be formulated as materialism or as spiritualism of which materialism is the more important partly for reason of the consistency of the system and partly for the number of its followers. Materialism (spiritualism) is the ontological view that extended matter (thinking substance) is the only existing thing. Both forms of the monistic hypothesis meet with one common difficulty, but against materialism there is an argument

available to which spiritualism is not open. This argument may be called the epistemological argument against materialism. It is based on the fact, that what is immediately given to us is consciousness and that matter is known to us merely by our ideas. To reduce mental life to matter is therefore an explanation *per ignotius*.*

Every monistic system is called upon to explain how a substance may produce phenomena so fundamentally different in their qualities. Materialism can explain everything but thought; spiritualism can explain everything but matter. The *materia cogitans* can not be reduced to the *materia extensa* nor *vice versa*. Materialism and spiritualism are equally open to this argument which may be called the ontological argument against monism. This argument is greatly used in the different refutations of materialism,† and Leibnitz's example of the mill‡ is very likely the first exhibition of it. No new argument against materialism was found in spite of great effort and, as F. A. Lange remarks,¶ it is always the same hit, the impossibility to reduce the psychic to the physis, which deals a knock-out blow to materialism.

*The trend of this argument goes back to KANT *K. d. r. V.*, Werke, ed. Hartenstein, Vol. 3, pp. 606, 607 (omitted in the second edition.) The full statement of this subtle argument is due to SCHOPENHAUER, *W. a. W. u. V.* Werke, ed. Griesebach, Vol. 1, p. 62 sqs. "Der Materialismus ist also der Versuch das unmittelbar Gegebene aus dem mittelbar Gegebenem zu erklären." This argument was adopted also by Riehl, Schuppe, Bergmann, Adickes and others; see L. BUSSE, *Geist und Körper, Seele und Leib*, 1903, pp. 15-17, who states the argument in full and gives references to further literature.

†Materialism has two forms: 1.) The psychic is a particular kind of matter; 2.) the psychic is a type of motion. The latter view may be formulated more cautiously in this way: The effects of the psychic are equivalent to motion. It seems that this formulation avoids the ontological argument but in this formulation, materialism has lost entirely its ontological character taking the phenomenistic point of view that a force is known to us merely by its effects.

‡One finds this argument very frequently. The last statement is that of C. L. HERRICK, *The Nature of the Soul*, *Psych. Rev.* Vol. XIV, 1907, p. 208, where the argument is attributed to Rabier.

¶F. A. LANGE, *History of Materialism (Engl. Transl.,)* 1881, Vol. III, p. 329. For the critique of other arguments against materialism L. BUSSE, *Geist und Körper, Seele und Leib*, 1903, pp. 50-61.

The dualistic hypotheses are in turn confronted with the necessity of explaining how two substances which are so fundamentally different can influence one another. The relation of these two substances is the problem to be explained by dualism; the intrinsic difficulties of this view are so great that the authors who profess these ideas introduce them, very frequently, as the one hypothesis against which the smallest number of arguments tells.* Such a view, even if full credit is given to all the arguments of the authors, has of course not the character of the most probable, but only of the least objectionable hypothesis, and it amounts almost to a renunciation of every theoretical intelligibility to introduce the incomprehensible right at the beginning of one's explanation.† The dualistic systems occur in the following three forms: 1.) Interactionism, 2.) occasionalism and 3.) parallelism, the first and third form having two different types.

*Prof. JAMES, for instance, introduces in his "*Principles of Psychology*" the notion of a soul in this way at the end of a long discussion of other views; the trend of Busse's argumentation is similar.

†The same objection holds against the view that one has to consider it as a *fact* that the psychic influences the physical, and vice versa, and that the incomprehensibility of such an influence, which makes this influence equivalent to a miracle, is no objection, "because a miracle that happens every day ceases to be a miracle." This view was taken by Stumpf and Jerusalem. This argument proves too much. It is for instance also available in defence of occasionalism, which is thoroughly acceptable if one is satisfied with resolving the events in an uninterrupted chain of miracles. The argument, however, misses the following point: The difficulty of the problem lies in the correct definition of notions which must be serviceable for the description of certain facts, about which is little doubt in so far the *quaestio facti* goes. The notions used until now do not serve this purpose of a correct and contradictionless description. In other problems one would dispose of notions which lead to contradictions and try new ones. If the notions which are available should not meet with better success one would suspect either that the problem is insolvable, or that our present means are not sufficient for the solution. In the first case one is satisfied with the demonstration that the problem cannot be solved and one leaves the problem aside, as it was done with the problems of squaring the circle and of constructing the perpetuum mobile. An example of the second possibility is the problem of n bodies, of which it was shown lately that our present means are not sufficient for a general solution. The peculiarity of the mind-body problem lies in its connection with other highly important questions, which make it desirable to solve the problem in one way or the other.

Interactionalism is found as the theory of an *influxus physicus* or as the theory of an *influxus psychicus* or as a combination of these two theories. The first type of the third form of the dualistic hypothesis is the pre-established harmony, and the second is psychophysical parallelism in the proper sense, the principle of which is that mental and physical phenomena are independent from each other, so that one of them cannot be reduced to the other, and that they go parallel without being in causal relation. Psychophysical parallelism was stated in many different ways, but the characteristic feature of parallelistic systems is the alleged impossibility to reduce psychical to physical phenomena or *vice versa* and the lack of causal relations between them. A certain form of parallelism (universal parallelism in Busse's terminology) approaches closely the view of the pre-established harmony. Parallelism is called upon to explain the intercourse of two "spirits," as e. g. in conversation, but this argument avails also against the other dualistic theories except interactionalism.

The dualistic systems meet with a peculiar difficulty which is caused by the application of the category of substance. This difficulty consists in the impossibility of demonstrating strictly the existence of other thinking beings, the term "thinking being" referring a subject which is the bearer of conscious states. There is no necessity for the assumption of conscious beings besides the one thinking individual, because what we perceive are phenomena belonging to the class of those which are attributed to the *materia extensa*, and only our own conscious states are immediately given to us. This view is known under the name of absolute idealism or transcendental egotism. The scholastic formulation of this view maintains that we perceive only the effects of conscious states and that the conclusion from the effect to the cause is not certain. This argument seems to be inevitable under the assumption of a substance which underlies the thinking process and the force and peculiarity of this argument is, perhaps, best characterized by Schopenhauer's remark that transcendental egotism remains an unconquerable position which, however, as a serious point of view can be found only in the madhouse. This much disputed argument seems to give a peculiar position to the mental states and it is not void of interest that it

may be applied to objects of every description. This becomes obvious at once when a question is raised like this: How do we know that there is magnetic substance in a magnet? A magnet is defined by the qualities of bodies in general and in addition to these by all those reactions which are characteristic for the magnetic state. There is nothing about a substance in the data and all we know about a magnetic body is exhausted with their description. In the same way we do not know of intelligence in a thinking individual except by its reactions which, however, are not as clearly defined as those characteristic for magnetism, and which are, furthermore, less stable. This lack of stability is the reason why the notion of substance is of so little use in the treatment of psychical phenomena and, no matter whether we regard with Mach the notion of a substance as a hypothesis introduced for the sake of explanation, or whether we consider it with Kant as one of those notions which we are bound to apply to experience—two views which are by no means so very widely different and which may be varied in detail considerably—the fact remains that the notion of substance, though very useful in physics, is of no practical avail in psychology. In physics the notion of substance leads to that of matter, as that which is extended, moveable and impermeable, but when applied to mental states it can be used only as the notion of an indefinite “something”, which is of no use. It is a very obvious idea to try how much one can do without this notion.

In order to avoid all these difficulties one may try to formulate the problem in such a way that it loses its ontological character, retaining only the general problem of finding relations between phenomena. Events are the only immediate datum of experience and one may try to find relations between them. The complete description of an event requires that all the events which are connected with it according to a definite rule are described. A mental state is connected not only with events of the class A, B, C, ... but also with events of the class a, b, c, ... which must not be omitted in a complete description. Psychophysical parallelism, thus, gets a very simple expression on this ground. The general principle of psychophysical parallelism is that the events A, B, C, ... are in a relation with the events

a, b, c, ... and that a mental phenomenon A is not explained before those events of the class a, b, c, ... are described with which A is in a definite relation. This principle does not make any definite assumption about the nature of this relation, but it is rather the expression of the fact that the phenomena of the classes A, B, C, ... and a, b, c, ... constitute together the realm of experience, in which we may look out everywhere for connections between the events. A practical consequence of this principle is that the investigation of the regularities of the connection between events, especially of events of the class A, B, C, ..., must not stop at any boundary inside the realm of experience. The existence of definite rules for the connection of events is a supposition of the possibility of constructing scientific systems, because it would not be possible otherwise to deduce an exhaustive system from a finite number of principles. The supposition that there are no disconnected events, i. e. wonders, in the entire realm of experience is the condition of a complete description of the phenomena. In the description of an element of the class A, B, C, ..., e. g. a color sensation, the problem of psychophysical parallelism is not exhausted with the answer to the question: Which are the phenomena of the class a, b, c, ... which are connected with this event according to a definite rule? The class of phenomena the description of which is required by this question contains among other processes those which take place in the optic nerves and in the occipital lobes under the influence of retinal stimulation. This answer shows in itself that this description is only a part of the answer to the more general problem which does not restrict the events to those of the second class, and which requires the description of all the events with which the first event is in relation. Events of the class A, B, C, ... have the same claim as those of the class a, b, c, ... and the view that psychology, i. e. the science which deals primarily with the events A, B, C, ..., should be resolved into a special science of the events a, b, c, ..., e. g. into brain physiology, is not essentially superior to the opposite view that the events a, b, c, ... are indifferent for psychology, although it may lead at present to a greater number of propositions, because the relations of the second class are better known. The statement that mental phenomena depend on physical phenomena is fre-

quently nothing else but an expression of the principle of psycho-physical parallelism. Sometimes, however, this statement is given the more definite meaning that spatially well defined groups of events of the class $a, b, c, \dots$ are in constant relation with certain events of the class $A, B, C, \dots$ so that the event A does not occur if $\dots m, n, o, \dots$ did not occur, and that $\dots m, n, o, \dots$ occur when A is observed. Such a statement contains a special law and is necessarily the product of observation. The best known example of a relation of this type is the so-called principle of cortical localization. The term "principle" is well chosen if this proposition is given the meaning that there exist such groups of events $\dots m, n, o, \dots$ for every event of the class $A, B, C, \dots$. This proposition is of higher generality than a result of observations can be and it deserves the name of a principle, because it lays down a rule for an important part of physiology. The term "principle of cortical localization," however, is not well chosen, if it is applied to that group of propositions which are the outcome of all the investigations along this line. This group of propositions is the expression of empirical facts and the term "empirical law" or "law" is more appropriate. This relation between events $A, B, C, \dots$ and events of the class $a, b, c, \dots$ seems to be very mysterious, if this relation is conceived as a relation between two fundamentally different substances, but this mystery disappears if the purely phenomenological point of view is taken. Both events stand on the same level and their relation is not more mysterious than that between any two others. The existence of definite relations between spatially well defined groups of physical phenomena and certain mental phenomena shows that it is by no means an impossible task, involving a contradiction, to establish relations between phenomena of the class $A, B, C, \dots$ and those of the class $a, b, c, \dots$.^{*} The contradiction only comes in when the phenomena are referred to substances which are fundamentally different. Treating the data of experience in a purely

^{*}An analysis of a very complicated group of sensation into its elements was given by E. MACH, *Analyse der Empfindungen*, 1902, pp. 32, sq. This author describes in this book as well as in his "*Thermodynamics*" and in his "*Mechanics*" the motives which may induce us to give to the objects a reality independent of our sensations.

phenomenological way without making use of the idea of substance shows that the barrier which separates the class A, B, C, ... from the class a, b, c, ... is not insurmountable, or rather that the problem of establishing relations between the events of these two classes is not essentially different from the problem of finding relations between other groups of phenomena.

The inadequacy of the substitution of relations between substances for the purely phenomenological relations between events is most apparent in the problem of the relation of the psychical to the physical. The view that all events are modifications of substances may be called the theory of mechanical or substantial causality; this view implies that substances are the ultimate reality of everything. The shortest expression of this view that substance is the factor undergoing changes is given in the sentence: All causing is effecting (Alles Wirken ist ein Bewirken).^{*} This view is eminently anthropomorphic, because it takes the will act as the type of a cause. The difficulty of conceiving of the relation between thinking substance and extended substance is by no means the only shortcoming of the notion of substantial causality. This view, in fact, leads to many other difficulties and contradictions as e. g. to the statement that every judgment must express a relation between substances, or between a substance and its *accidentibus*. The only type of judgments admissible under this supposition are those judgments in which the relation of a subject to its predicates are determined. This classification, however, does not provide for the so-called impersonal judgments to which this definition does not apply. This fact shows that the notion of substantial causality does not exhaust even those phenomena which have no direct bearing on the mind-body problem. It is, therefore, advisable, to see whether the notion of substantial causality cannot be superseded by another. We shall use the notion of relations between phenomena; causality, then, is conceived as a type of order between events.

^{*}The exponents of this theory are fairly numberless, and we quote only as an example C. A. STRONG, *Why the Mind has a Body*, 1903, p. 75, "A volition seems the very type of a cause." The most recent exposition of this view was given by Professor FRANK THILLY in his article "*Causality*," *Philosophical Review*, Vol. 16, 1907.

The term relation in its broadest signification has the meaning of a one-to-one relation between two elements; this notion if it is confined to quantities coincides with the notion of a mathematical function in the sense of Cauchy and Dirichlet. Such a function is a rule by which a value of the dependent variable is adjoined to every admissible value of the independent variable. The most general form of a relation is that of a non-uniform, discontinuous function. This definition is too general to be treated by our present means with advantage, and one restricts the investigations first to the uniform, continuous functions, and later on it becomes necessary to restrict it still further to the analytic functions. A continuous function may be given by a numerable, infinite set of conditions, but this is not the case of the general discontinuous functions. These functions are characterized by an innumerable, infinite set of conditions, what is the same as to say that they cannot be defined. In consideration of the fact that Fourier's analysis shows, how certain discontinuous functions may be represented by a sum of continuous functions one might have taken the view that these general functions may be represented, or approximated, by analytic functions. This view, however, would hardly have been a reasonable expectation, and it was made impossible lately by the discovery of Baire, that every function which can be represented as the limit of continuous functions must be a function with not more than punctual discontinuities. If it should happen that a function of more than punctual discontinuities occurs in nature, it would be not only impossible to represent it in the usual way but it even would be impossible to approximate it. An event of this type could not be described in the same terms as ordinary events. A classification of events of this type cannot be exhaustive and it is not possible to deduce systems of science from a limited number of propositions, because there is no warrant that one may not find at any moment an event which does not resemble in anything any element of previous experience.

*See BOREL, *Leçons sur la théorie des fonctions*; LEBESGUE, *Leçons sur l'intégration*; BAIRE, *Les fonctions discontinues*; PHILIP E. B. JOURDAIN *The Development of the Theory of Transfinite Numbers*, *Archiv der Mathematik und Physik*, Ser. III. Vol. 10, 1906, pp. 254-281.

It may be a question under which conditions systems of science are deducible from a limited set of propositions, using only a limited number of fundamental ideas or notions, but there is no doubt that such systems cannot possibly be exhaustive of a certain realm of experience if non-uniform, discontinuous functions are admitted. Winter* has tried to argue against the possibility of deriving a system of science from a limited number of fundamental notions by means of the fact that mathematics in its natural progress constantly introduces new notions, the definition of which is apparently more or less arbitrary. Such is e. g. the origin of the elliptic functions which are obtained by increasing the order of a function in a certain integral which leads to known functions by the unit. Against this argumentation one may say that the way pointed out by Winter is the way of the discovery of new functions, but that every function can be derived by starting from the series which gives the definition of the function. This series is constituted by no other but known operations, so that the new function is expressible by an algorithm which contains no other but elementary operations of which there is only a limited number. The introduction of new functions, therefore, is not an argument against the possibility of a system being built up on a limited number of notions, as long as the new functions can be reduced to the forms of relations already known. This possibility is not given for functions which belong to the general type of a function in the sense of Cauchy and of Dirichlet.

We have introduced above the distinction between regularity and uniformity of events. Uniformity is the more special case of regularity, because it may be that the rules of the succession of phenomena vary in such a way that different rules of succession are adjoined to the same phenomenon at different times. The type of order which is applied to the study of natural phenomena is the one of exclusive uniformity, requiring that the same group of events under given conditions is always followed by a certain other group. These relations are most commonly found in mechanics, and for this reason one frequently calls this type of relation

*M. WINTER, *Sur l'introduction logique à la théorie des fonctions*, *Revue de Métaphysique et de Morale*, Vol. 15, 1907, pp. 205-209.

mechanical causality. The simplest example of events of this kind is the push and pull of one mass on another mass, and the task of constructing a system of science in which the events are causally determined was frequently formulated in this way, that every event must be analyzed into motions of masses which are moved by the push and pull of other masses and which have no power of their own to influence their motion (*vis a tergo*).* The beauty of this conception is that the influence of one mass on the motion of another mass seems to be immediately intelligible by the analogy of the influence of our body on its surroundings. The consideration of mechanical phenomena, however, leads only to one class of relations, and it was seen at a very late time that the problem thus restricted excluded all relations which do not belong to a comparatively small class. The class of relations which are admitted for the description of natural phenomena are the so-called analytic functions. The view that all phenomena are under the principle of mechanical causality requires that natural phenomena are described merely in terms of analytic functions. These functions have, besides some other minor properties, the following distinguishing peculiarities: They are single valued, continuous, differentiable, they can be developed into a power series, they admit of an analytic continuation, and they are solutions of differential equations. These peculiarities make the analytic functions so precious for the study of natural phenomena. The conditions in infinitesimally small intervals of time and space are so simple, that we can express them with comparative ease by differential equations, the integration of which leads necessarily to analytic functions. Having established such a relation in no matter how small an interval of time and space, we can determine and foresee every phase of the later de-

*The value of this view is the question at issue in the discussions between the adherents of the energetic view and those of the atomistic hypothesis. The denial of the indispensability and of the adequacy of the atomistic hypothesis is the contention of the energists (E. MACH, *Die Mechanik in ihrer Entwicklung*, p. 486 "Dass alle Vorgänge mechanisch zu erklären sind halten wir für ein Vorurtheil,") whereas the adherents of the atomistic hypothesis point out that the energetic view does not show a success in the explanation of natural phenomena nearly equal to that of the atomistic hypothesis.

velopment of the process, because the function which describes this process has an analytic continuation. The course of events, which is the course of nature, is uniquely determined so far as it is characterized by analytic functions. Events which are characterized by analytic functions are completely under our control and none of the details of such events can escape detection. Physics deals exclusively with processes which may be described by analytic functions and the success of this science suggested the view that all events, if properly analyzed, must lead to this type of relations.* If it is true that there are in nature no other but analytic functions, then there exists indeed the possibility of exhausting the description of the world by the future progress of science. The observation of an event in no matter how small an interval of time and of space permits to predict the future and

*We quote only B. RIEMANN, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, *Werke*, ed. WEBER, 1892, p. 237: "Durch die Arbeit Dirichlet's ward einer grossen Menge wichtiger analytischer Untersuchungen eine feste Grundlage gegeben. In der That für alle Fälle der Natur, um welche es sich allein handelt, war sie vollkommen erledigt, denn so gross auch unsere Unwissenheit darüber ist, wie sich die Kräfte und Zustände der Materie im Unendlichkleinen ändern, so können wir doch sicher annehmen, dass die Functionen, auf die sich Dirichlet's Untersuchung nicht erstreckt, in der Natur nicht vorkommen." The functions which Dirichlet's investigation does not cover are the functions with an infinite number of maxima and minima in a finite interval, and the supposition that there are in nature no other but analytic function comes out clearly in Riemann's words. The impossibility of the occurrence of functions with an infinite number of maxima and minima in the description of the motion of a point can be seen in this way. A body which moves along a curve can approach a point in the neighborhood of which there is an infinity of maxima and minima only with always decreasing velocity, and a movement from this point is impossible (See A. KOEPKE, *Differentirbarkeit und Anschaulichkeit von Functionen*. *Math. Annalen*, Vol. 29, 1887, pp. 137-140.) The reason for the requirement of a differentiable function is that a motion which is characterized by a function without a derivative would have to take place without a definite velocity or acceleration. Some of the modern writers have introduced differentiability as a specific requirement, e. g. HELMHOLTZ, *Vorlesungen über die Dynamik diskreter Massenpunkte* (*Vorlesungen über theoretische Physik*, Vol. 1, 1898,) p. 7, sq.; L. BOLTZMANN, *Vorlesungen über die Prinzipie der Mechanik*, 1897, pp. 10-13, who says that differentiability ought to be introduced as one of the requirements of mechanics.

to state every event of the past. The events at any particular point of time and space depend in a very complicated but perfectly definite way on all the other events, and it is merely a question of ability how much of the world's course in general one may be able to deduce from one single fact no matter how trivial it may be. The perfect knowledge of all these relations would be the realisation of Laplace's mechanical ideal, which is only a consequence of the supposition of the exclusive occurrence of analytic functions in nature i. e. of the absolute intelligibility of all natural phenomena. Special mechanical principles, as e. g. the theorem of the conservation of energy, depend on the principle of general causality formulated as the exclusive occurrence of analytic functions in nature, and they do not hold generally if other functions are admitted.* This view that all events are characterized by analytic functions and that everything can be treated by the same scientific method was called materialism by Alexejeff,† but since this term is used for so many widely different notions and

*As long as no sets of necessary and sufficient axioms of mechanics are constructed it is very hard to tell whether a proposition has the character of a principle or of a law. SIGWART (*Logik* 2, ed. 1893 Vol. 2, p. 644) says he cannot convince himself that Galilei's inertia and Newton's universal attraction are necessary consequences of the principles, so that he would be inclined to regard them as natural laws. This may be true for the special form of the Newtonian potential but it is not equally obvious for the law of inertia which was frequently formulated as a negative expression of the principle of causality. For literature on the problem of inertia see L. LANG *Das Inertialsystem vor dem Forum der Naturwissenschaft, Phil. Stud.* Vol. 20 1902, pp. 1-71. The proposition of the conservation of the energy was regarded by Helmholtz as a consequence of the causalistic conception of the world, whereas other investigators try not very successfully to look at it as a datum of experience, e. g. ERICH BECHER, *Das Gesetz von der Erhaltung der Energie etc.* *Ztschr. f. Psychologie*, Vol. 46, 1907, p. 98. On the principle of the conservation of energy see W. WUNDT, *Physiologische Psychologie*, 5, ed. Vol. 3, 1903, p. 634. A discussion of the views on this question as advocated by different authors is given by L. BUSSE, *Geist und Körper, Seele und Leib*, 1903, p. 119, 124-126.

†W. G. ALEXEJEFF, *Die arithmologische und wahrscheinlichkeitstheoretische Kausalität etc.*, *Ztschr. f. Philosophie u. Pädagogik*, Vol. 14, Nov. 1905. pp. 50-55 speaks of "Materialismus oder die rein physikalische Betrachtungsweise des Geistigen." We use only the second term of the two suggested by Alexejeff.

since it is primarily a term for an ontological view, it may seem best to find another name for it. The term "mechanistic view" would be appropriate, if it were not frequently used for the designation of the atomistic hypothesis, and it is, perhaps, best to call this view after physics, the science which most successfully uses it. By the term "physical point of view" we intend to designate the view that there are no other but analytic functions to be found in nature.*

The problem of the description of mental life calls for the establishment of the laws of the succession of these phenomena and the broadest solution of this problem would be to find a one-to-one relation between every moment of time and a certain element of the class A, B, C, ..., but leaving the question open whether this relation has the character of an analytic function or not. The way of proceeding may be illustrated by the following example taken from physical science. The temperature of a liquid can be raised by bringing it into contact with a body of higher temperature. The temperatures of the body and of the liquid vary until the flow of heat comes to rest at a certain time, but in every moment the liquid has a certain temperature as well as the body, so that there is a one-to-one relation between the moments of time and the temperatures of the liquid. The accurate description of this process is supplied by the mathematical relation which shows how the temperature of the liquid depends on time. A similar example of psychology would be found in the description of the succession of the mental states under the

*It happens not unfrequently that the supposition that an event must have an analytic expression is superseded by the other supposition that it must be a simple expression. An example may be found in B. A. GOULD, *Use of the Sine-formula for the Diurnal Variations of Temperature*. *Amer. Jl of Science*, Vol. 119, 1880, who tries to show (p. 217) the importance of finding general expressions, because an expression which can represent n observations but depends on a number of parameters smaller than n is a real natural law. A discussion of this view may be found in H. BURKHARDT, *Entwicklungen nach oscillierenden Reihen*, *Jahresbericht d. deutschen Mathematiker Vereinigung*, 1902, pp. 249, 250. Burkhardt's opinion that this view is characteristic of the American for Anglo-American conception of natural science is perhaps not entirely justified. The conception that a natural law must be a simple law leads of course to a very flat rationalism.

influence of an incoming stimulus; the complete solution of this problem would give a one-to-one relation between every moment of the time under consideration and a certain mental content. There is no doubt about it that mental phenomena are functions of time and the difficulty merely consists in defining these functions. We dismiss from the start the supposition that the nature of this relation is known before hand. It seems to be very obvious to draw the following erroneous conclusion.

There exists a one-to-one relation between the elements A, B, C, ... and the moments of time—no specific hypothesis being made about the nature of this relation—but there exists also a definite relation between the moments of time and the elements a, b, c, ... One supposes that the latter relation must belong to the causal type i. e. that it must be expressible in analytic functions. Instead of merely concluding that there must be a relation of some kind between the elements A, B, C, ... and the elements a, b, c, ..., one may try to specialize these relations to the type of analytic functions. This conclusion is of course unwarranted, because the combination of an analytic function with a non-analytic function does not give an analytic function. We may illustrate the far reaching importance of this supposition by Mr. Montague's theory of the specious present.* This author starts from the observation that physical and psychical events are both functions of time, and he immediately proceeds to form the derivative of the function which gives the subjective phenomena as depending on the objective phenomena; to this quantity he

*W. P. MONTAGUE, *A Theory of Time-Perception*, Amer. Jo of Psychology. Vol. XV, 1904, pp. 1-13. Mr. Montague came very near to considerations like those expounded in this chapter, but instead of inquiring into the principles of the problem he unfortunately undertook to solve a very special and very difficult question. Some of the weaknesses of Mr. Montague's argumentation are shown in the author's note on "*The Application of Calculus to Mental Phenomena*," *The Jo of Phil. Psych. and Scient. Methods*, Vol. II, 1905, pp. 16-18, but only a general allusion is made to the fact that the weaknesses of Mr. Montague's theory are the necessary outcome of a certain definite hypothesis on the relation of the physic and the psychic.

gives a certain psychological interpretation.* Thus the supposition crept in that the dependence of mental on physical events is given by differentiable functions. This supposition excludes at once a very wide range of possibilities, and it contains the definite statement that the type of order of the events of the class $A, B, C, \dots$ is the same as that of the events $a, b, c, \dots$. The reason why this rather obvious fact could be overlooked lies in the peculiar formulation which the author gives to his problem. He introduces both types of events as functions of time and concludes that the events $A, B, C, \dots$ must be functions of $a, b, c, \dots$. There is nothing to say against this conclusion, if the hypothesis is not made that there exists a derivative of this function. If one makes the supposition that the events of the class $A, B, C, \dots$ and those of the class $a, b, c, \dots$ are analytic functions of time, one must necessarily accept the conclusion that the events $A, B, C, \dots$ are analytic functions of the events $a, b, c, \dots$. This supposition certainly had not been made, if it had been seen what the bearing of the hypothesis is that mental events are analytic functions of physical phenomena.

Physics leads necessarily to analytic functions for the description of natural phenomena, because they are the solutions of differential equations and it is obvious that all the problems which deal with events which cannot be described by them must escape notice. This caused the question whether the description of phenomena as supplied by physics is exhaustive, and on what ground the supposition rests that there are no other but analytic functions in nature. This question was raised by F. Klein,† who gave a very ingenious answer which is based on the distinction between functions which are given empirically and their ideal mathematical expression, a distinction which this author had introduced previously.‡ The empirical representation of a func-

*Mr. Montague's interpretation of the derivative did not remain uncontradicted; a discussion of this topic may be found in E. B. HOLT, *Jo of Phil. Psych. and Scient. Methods*, Vol. 1, 1904, pp. 320-323 and Mr. MONTAGUE's answer in the same journal, Vol. 1, 1904, pp. 378-382.

†F. KLEIN, in his lectures of 1901, *Vorlesungen über Differential und Integralrechnung, eine Revision der Principien*, ed. 1907, pp. 129-139.

‡F. KLEIN, *Über den allgemeinen Functionsbegriff und dessen Darstellung durch eine willkürliche Curve*, *Math. Ann.* Vol. 22, 1883, pp. 249-259.

tion is never an ideal curve but an area of finite extension and what is given empirically are neither analytic nor non-analytic functions, but only approximations. The distinction between analytic and non-analytic functions refers merely to their idealized mathematical representation. This solution cannot be discussed here and also another solution given by J. Boussinesq can be but briefly mentioned.* Boussinesq starts from the view that the exact description of a phenomenon is given by its differential equation with which its integral is equivalent, because purely logical processes are needed for finding the integral of a differential equation. These solutions are given by analytic functions which determine the process everywhere except at certain points, the branch points, where the choice between different courses comes in. The course of events is not defined at these points and it is there, at the branch points, that Boussinesq looks out for the manifestations of mental life, more especially of free will, which manifests itself by the undetermined choice between equal possibilities represented by the branches of the curve. This view has not found many followers and it stands out more as a curious example of the ingenuity of its promoter.

The third attempt at a solution of our problem has a direct bearing on the question treated here, and this view is formidable by the number of distinguished followers it has found. The starting point for these considerations is found in the fact that certain phenomena which are the product of human will-decision show a remarkable stability in their averages, if they are taken in large groups, no matter how much chance may have influenced the single decisions. These results of demography and statistics of

*J. BOUSSINESQ, *Conciliation du véritable déterminisme mécanique etc. Recueil de la Société des Sciences de Lille*, Vol. VI, 1878 and C. R. Vol. XCIV, p. 208. Boussinesq treats the same problem in his book "*Application des potentiels à l'étude de l'équilibre et du mouvement*," 1885, pp. 699-704, where he attempts to show that functions which have no derivative can be treated in the same way as other functions by considering functions which differ from them only in a very slight degree. In this attempt he approaches Klein's conception of a "*Funkionsstreifen*" by making it a matter of preference which one of a certain group of functions, which differ from a given function by less than a certain quantity, one will treat.

morals have always aroused great interest and were frequently used as arguments in favor of universal mechanical causality against free will. Instead of drawing these conclusions one must stop considering the conditions under which events, which are taken in large groups, can possibly show regularity in the mean values. In regard to this question there exists a famous proposition due to Tchebitcheff, one of the promoters of the view in question, that the chief condition for the applicability of mean values is that the single events must be independent from each other. The results admit of a correct interpretation if the events are independent from each other, but if they are not independent i. e. if they are causally connected, one cannot apply the theorems of the calculus of probabilities. From the fact that we find regularity in large groups of human actions we must conclude that they are independent i. e. that they are not in a causal relation, and since all events of the causal chain are inter-connected we must conclude that will decisions are not causally necessitated. The law of causality is, therefore, not the only one and not all events of nature are characterized by analytic functions. In this way the old argument against free will is turned into an argument for it. The argumentation as expounded is due to the so-called Moscow school of idealism as represented by P. L. Tchebitcheff, Nekrassow, W. G. Alexejeff, and the head of the school N. W. Bugajeff. These men have also materially promoted the study of the "half-analytic" functions which originate by the combination of analytic with non-analytic functions, and in which these authors see a more adequate expression of the relation of the physic and the psychic, and important works on the theory of probabilities are due to them. Alexejeff has the merit of making these investigation generally accessible by several German papers.*

*W. G. ALEXEJEFF, *Über die Entwicklung des Begriffes der höheren arithmologischen Gesetzmässigkeiten in Natur- und Geisteswissenschaft*, *Vierteljahrsschrift f. wiss. Phil. u. Soz.*, Vol. 28, 1904, pp. 72-93; "N. W. Bugajew und die idealistischen Probleme der Moskauer mathematischen Schule," the same journal, Vol. 29, 1905, pp. 335-367; and "Die arithmologische und wahrrscheinlichkeitstheoretische Causalität etc.," *Ztschr. f. Philosophie u. Pädagogik*, Vol. 14, (2.) 1906, pp. 50-55.

This argument proves too much, because it applies to all kind of random events among which there are many classes of events which doubtlessly are causally necessitated. There are events which we can submit successfully to the treatment by the calculus of probabilities, although the single events are causally necessitated or even logically determined. There cannot be any doubt in these cases that an independence of the single events does not exist in any absolute sense of the word, but large groups of events comply with the requirements of the calculus of probabilities in a very high degree. Let us take the following examples. Whether high water arrives at London Bridge in the first, second, third or fourth quarter of an hour of the morning is the very type of an event which is causally necessitated. The single events depend entirely on physical conditions, but the results of a long series of observations conform, nevertheless, with the requirements of the calculus in a very high degree.* The following example of events which comply almost perfectly with the requirements of the calculus of probabilities, although they are logically determined, is due to Bruns. The distribution of zeros at the last place of the logarithms in the columns (of 60 numbers each) of Vega's *Thesaurus Logarithmorum* shows that the numbers of those columns in which zero occurs 1, 2, ... 60 times is very nearly such as it ought to be, if the occurrence of the numbers at the last place were entirely a matter of chance.† It is quite obvious that the

*F. Y. EDGEWORTH, *The Law of Error, Part II*, *Transactions of the Cambridge Philosophical Society*, Vol. 20, 1905, p. 128, 129.

†H. BRUNS, *Wahrscheinlichkeitsrechnung und Kollektivmasslehre*, 1906, pp. 8, 279 sqs. Bruns had chosen this example on purpose, because in the case of the occurrence of the different numerals at certain places of the logarithms one cannot possibly speak of chance in the common sense of the word. The observation that tables of logarithms are material for the study of chance events was made by GAUSS, *Einige Bemerkungen zu Vegas Thesaurus Logarithmorum*, *Werke*, Vol. III, p. 260. Gauss remarks that the last decimal of the logarithms of the tangent cannot be equal to the last decimal of the difference of the logarithms of the sine and cosine, if the deviations of the logarithms of the sine and of the cosine from the correct values have not the same sign, and if their sum is greater than $\frac{1}{2}$. The probability of this event is $\frac{1}{2}$. Gauss found by counting over the tables of Köhler that this event happened in 250 out of 900 cases.

occurrence of a certain numeral at the tenth place of the logarithms of consecutive numbers is not an event which has random character in an absolute sense of the word, since we can determine the event in every case by the rules of arithmetic. The fact is that randomness of the events, which can be submitted to the treatment by the calculus of probabilities, consists in the absence of a knowable law. This, as a matter of fact, is the conclusion from the test whether a certain group of events has random character or whether it has not, that the agreement with the rules of the calculus of probabilities indicates the absence of a recognizable law, and the lack of agreement with the results of the calculus indicates the presence of a constant influence which may be investigated. This type of probability, which may be called randomness by excessive complication, has been recognized already by Kepler, as it was pointed out recently by Bruns.* It is exactly this type of probability which we use for the working out of the results of our experiments on lifted weights. Every single event is causally necessitated, but its conditions are so complicated that we must be satisfied with this very general type of relations which form the propositions of the calculus of probabilities. A mathematical probability is defined as a fraction the numerator of which gives the number of cases which are favorable to the event, and the denominator of which gives the total number or possible cases. The outcome of every single case must be well defined. We have seen above that the numbers of relative frequency for the judgments of different types on the comparison of two weights have the formal and material character of prob-

*H. BRUNS, *l. c.* p. 7. The place in question is J. KEPLER, *De stella nova in pede Serpentarii*, Opera, ed., FRISCH, Vol. II, p. 714 "Improvidi sunt, qui hos (tesseractum jactus) plane fortuitos, hoc est *ἀνακρίους* esse putant: sin autem suum casum omni causa privant, nondum ejus exemplum dixerunt in vesseris. Quare hoc jactu Venus cecidit, illo canis? Nimirum lusor hac vice tessellam alio latere arripuit, aliter manu condidit, aliter intus agitavit, alio impetu animi manusve projecit, aliter interflavit aura, alio loco alvei impexit. Nihil hic est, quod sua causa caruerit, si quis ista subtilia possit connectari." The example of the acetarium on the previous pages of Kepler serves as illustration against certain ancient systems of philosophy that all possible combinations of events must be exhausted in an infinite interval of time.

abilities. The number of cases which are favorable to the event, e. g. to the formation of the judgment "heavier", is represented by a finite or infinite number of groups of conditions each one of which leads necessarily to the formation of the judgment that the second weight is heavier. The number of possible cases is represented by the totality of conditions in which a judgment is given according to the rules of the experiments. The ratio of these two numbers gives the probability in question. Only under the supposition that there exists a definite (finite or infinite) number of possibilities each one of which leads necessarily to a certain result can we form the notion of the probability of a "heavier"-judgment. We conclude from the fact that the notion of a mathematical probability can be successfully applied to physical events, that they have very nearly the character of those events which we treat in the system of idealized propositions which we call the calculus of probabilities. The proposition that a certain well defined group of physical events has the character of random events in the mathematical sense of the word warrants the application of the rules of the calculus to these events. This statement plays the same role for this group of events as the statement that physical space is a three-dimensional Euclidean manifoldness plays for the application of ordinary geometry to physical space. The conformity of events with the rules of the calculus of probabilities is so far from being an argument for the lack of causation of these events, that it is an argument for uniform causality. No statement about the future outcome of observations were possible, if every single event were not causally necessitated.*

*This conception is a consequence of basing the idea of probability on the hypothetical judgment, as it was done by SIGWART, *Logik*, Vol. II, p. 305-320. For the logical details the reader must be referred to this book and to the treatises of K. STUMPF, *Über den Begriff der mathematischen Wahrscheinlichkeit*, Ber. d. bayr. Ak. (Phil. Kl.) 1892; J. v. KRIES, *Die Principien der Wahrscheinlichkeitsrechnung*, 1886; H. BRUNS, *Wahrscheinlichkeitsrechnung und Kollektivmasslehre*, 1906, and to the repeatedly quoted works of E. CZUBER. A different view was expressed lately by Mr. GOMPERZ, *Über die Wahrscheinlichkeit der Willensentscheidungen*, *Sitzungsberichte d. Kaiserlichen Akademie der Wissenschaften zu Wien, Phil. Hist. Kl.* Vol. 149, III. Abh. who tried to find a new conception of free will and began by showing that the sta-

The process of finding the numerical values of these probabilities is perfectly well defined and stands on the same level with other empirical observations. The theorem of Bernoulli gives the limits of exactitude of such observations, and it depends on our will to make our results as exact as we choose. The empirical datum of a determination of a probability is a realm inside of which we may expect the result with a given probability. The limits of the exactitude of the observation can be made smaller than any given quantity by increasing the number of observations. These observations may be made for different intensities of the comparison stimulus. The result of these experiments is that a higher probability of a "greater"-judgments corresponds to higher intensities of the comparison stimulus, that greater probabilities of "smaller"-judgments correspond to smaller intensities of the comparison stimulus, and that the probabilities of the "equality"-judgments increase at first and decrease after having attained a certain maximum. These empirical data do not differ essentially from the results of any other experimental investigation, which show how certain quantities depend on the variation of another quantity. These data are subjected to a process of idealization, the first step of which is to assume that an absolutely exact observation would lead to a definite numerical value and an hypothesis is made which these values are. In

tistical regularities do not prove anything in favor of causal necessitation of the single events. "If one were to cast a die 60,000 times every year and if there were every year among the results approximately 10,000 aces, nobody will conclude that in every single case the ace had to come out necessarily." This view, which Mr. Gomperz does not support by arguments, is refuted by the examples mentioned above, where events about the causation of which there cannot be the slightest doubt comply with the requirements of the calculus of probabilities. Mr. Gomperz's papers is marred by some slips ("an almost infinite probability" l. c. p. 13) which indicate that the author is not intimately acquainted with the theory of probabilities, a supposition which is supported by the fact that a "mathematical friend" had to give the formula p. 16 of his paper, which is easily found by calculating the point of intersection of

the two eclipses $(\frac{x^2}{a}) + (\frac{y^2}{s_1}) = 1$ and $(\frac{x-a^2}{a}) + (\frac{y^2}{s_2}) = 1$ and putting $\frac{s_1}{s_2} = q$

Compare the review of Mr. Gomperz's paper in *Ztschr. f. Psychologie*, Nov 1906 Vol. 43, p. 318.

the case of observations on probabilities the result is determined by the theorem of Bernoulli, in empirical sciences this result is given as a rule by the most probable value as determined by the method of least squares. The next step of idealization refers to the rule of dependence between the variations of the quantities observed. The observations on a "greater"-judgment show that these probabilities increase with the intensity of the comparison stimuli. This observation holds only for the finite set of observations which were really made. One interpolates between the elements of this finite set of observations the elements of an infinite set which is of the order of the continuum. The elements of the infinite set are supposed to follow the same law as the observed elements. This supposition is based on the methodological principle mentioned above that unknown phenomena are supposed to follow the same law as observed phenomena of the same class, as long as no contrary instances are known. There is no difference between our treatment of psychological observations and the methods by which physical observations are treated, and we may say in general that the mathematical representation of empirical observations is nothing else than an idealization of experience.

APPENDIX

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TABLE 1.
VARIATIONS OF THE WEIGHTS USED IN THE EXPERIMENTS (IN MILLIGRAMS)

Number	Weight (in grams)	February 23, 07.	March 18, 07.	March 25, 07.	April 8, 07.	April 15, 07.	April 22, 07.	April 29, 07.	May 6, 07.	Sum of variations.
1	100	- 5	- 5	- 5	- 9	- 6	- 4	- 1	- 8	19
2	100	+ 2	+ 2	+ 2	+ 2	+ 3	+ 4	+ 5	+ 5	3
3	100	- 3	+ 3	+ 5	+ 2	- 7	- 4	- 6	- 2	31
4	100	—	—	—	0	0	- 1	0	- 1	4
5	100	0	- 4	- 4	- 3	- 4	- 3	- 1	- 3	11
10	100	- 4	- 3	- 6	- 7	- 8	- 2	- 4	- 6	16
11	100	- 4	- 4	- 7	- 3	- 5	- 3	- 4	- 7	15
12	100	- 4	+ 4	- 5	- 5	- 5	- 6	- 4	- 9	25
6	104	—	- 4	- 6	- 6	- 8	- 7	- 6	- 8	8
7	108	—	- 4	- 6	- 6	- 9	- 9	- 8	- 10	8
20	92	—	- 3	- 3	- 3	- 7	+ 1	- 7	- 1	26
21	96	—	- 3	- 3	- 2	- 5	- 3	- 9	- 5	16
51	84	—	+ 2	+ 3	+ 3	+ 2	+ 5	- 3	+ 5	21
52	88	—	- 4	- 4	- 4	- 5	- 1	- 10	- 5	19
23	90	—	+ 13*	+ 1	+ 1	0	- 2	- 6	- 3	11
24	94	—	- 4	+ 14*	+ 6	+ 6	+ 4	- 4	+ 2	40
25	98	—	+ 14*	- 2	+ 2	- 1	- 2	+ 2	0	16
26	102	—	+ 3	+ 4	+ 5	+ 5	0	+ 4	+ 5	12

*Corrected.

TABLE 2.

Date of observation	I	II	III	IV
April 26, 07	100.308	104.364	103.003	114.227
" 29, "	100.423	104.488	103.434	114.781
May 2, "	100.299	104.360	103.159	114.709
" 6, "	100.357	104.439	103.357	114.747
" 10, "	100.654	104.704	104.280	115.342
Observed variations	+ 0.115	+ 0.124	+ 0.431	+ 0.554
	- 0.124	- 0.128	- 0.275	- 0.072
	+ 0.058	+ 0.079	+ 0.198	+ 0.038
	+ 0.297	+ 0.265	+ 0.923	+ 0.595
Sum of variations	0.594	0.596	1.827	1.259

TABLE 3.
FREQUENCIES OF THE DIFFERENT CLASSES OF JUDGMENTS. SUBJECT I.

Series	84			88			92			96			100			104			108																
	T	hg	lg	h	l	1	T	hg	lg	h	l	1	T	hg	lg	h	l	1	T	hg	lg	h	l	1											
I 2a	4	2	2	0	46	17	2	15	1	32	20	9	11	4	26	26	16	10	8	16	34	28	6	13	3	8	7	1	42	0	5	5	0	44	1
IV 1a	3	1	2	0	47	5	0	5	0	45	10	2	8	7	33	24	16	8	19	7	22	22	0	26	2	2	2	0	47	1	5	4	1	45	0
IV 2a	4	1	3	0	46	7	3	4	0	43	24	6	18	1	25	33	15	18	2	15	34	29	5	8	3	2	1	47	0	3	3	0	47	0	
I 1	4	0	4	0	46	7	3	4	3	40	29	15	14	6	15	19	9	10	10	12	22	18	4	21	7	7	2	5	43	1	3	2	1	47	0
I 2	2	0	2	0	48	2	0	2	1	47	22	6	16	6	22	16	7	9	13	21	32	22	10	11	7	6	4	2	44	0	2	2	0	48	0
III 1	5	3	2	0	45	4	1	3	0	46	18	7	11	5	27	21	9	12	6	23	23	16	7	21	6	5	3	2	44	1	4	2	2	46	0
III 2	2	0	2	0	48	5	1	4	1	44	12	3	9	3	35	20	10	10	14	16	8	7	1	31	11	5	5	0	44	1	1	0	48	1	
IV 1	3	0	3	0	47	7	2	5	3	40	7	4	3	8	35	20	9	11	14	16	16	10	6	29	5	2	2	0	46	2	1	0	1	48	1
IV 2	1	0	1	1	48	2	0	2	0	48	7	4	3	0	43	20	9	11	5	25	18	14	4	26	6	3	3	0	47	0	0	0	50	0	
Σ	28	7	21	1	421	56	12	44	9	385	149	56	93	40	261	199	100	99	100	151	209	166	43	186	55	41	30	11	403	6	24	19	5	423	3

TABLE 4.
FREQUENCIES OF THE DIFFERENT CLASSES OF JUDGMENTS. SUBJECT II.

Series	84			88			92			96			100			104			108																
	T	hg	lg	h	l	1	T	hg	lg	h	l	1	T	hg	lg	h	l	1	T	hg	lg	h	l	1											
I 2a	3	2	1	1	46	4	2	2	1	45	9	6	3	4	37	10	6	4	11	29	17	16	1	22	11	7	6	1	31	12	3	1	2	47	0
IV 1a	0	0	3	47	7	3	4	0	43	8	5	3	6	36	13	10	3	13	24	17	11	6	22	11	8	2	35	7	6	5	1	42	2		
IV 2a	0	0	1	49	6	3	3	0	44	7	5	2	4	1	12	3	9	15	23	9	5	4	29	12	1	0	43	6	5	4	1	45	0		
I 1	5	2	3	2	43	3	0	3	2	45	7	5	2	10	33	10	4	6	17	23	13	6	7	24	13	3	1	2	38	9	3	2	1	47	0
I 2	1	0	1	0	49	8	3	5	1	41	8	3	5	8	34	10	7	3	17	23	13	8	5	25	12	5	3	2	39	6	6	3	3	42	2
III 1	2	1	1	0	48	6	4	2	1	43	12	7	5	3	35	14	9	5	15	21	9	3	6	21	20	2	1	1	47	1	1	0	1	46	3
III 2	3	1	2	1	46	6	3	3	4	10	6	4	4	36	15	7	8	16	19	14	7	7	27	9	2	1	1	48	0	5	2	3	45	0	
IV 1	4	2	2	1	45	7	3	4	1	42	14	7	7	3	33	21	13	8	11	18	11	5	6	34	5	7	4	3	43	0	4	2	46	0	
IV 2	2	0	2	1	47	4	2	2	2	44	10	8	2	10	30	11	6	5	17	22	5	1	4	34	11	5	1	4	43	2	3	1	2	47	0
Σ	20	8	12	10	420	51	23	28	11	388	85	52	33	50	315	116	65	51	132	202	108	62	46	238	104	40	24	16	367	43	36	20	16	407	7

TABLE 5.
FREQUENCIES OF THE DIFFERENT CLASSES OF JUDGMENT. SUBJECT III.

Series	84			88			92			96			100			104			108															
	T	h	l	T	h	l	T	h	l	T	h	l	T	h	l	T	h	l	T	h	l													
I 2a	0	0	0	50	2	1	3	45	11	8	3	12	27	11	10	1	27	12	9	5	4	26	15	3	2	1	46	1	1	0	1	46	3	
IV 1a	2	2	0	48	3	1	2	5	42	10	9	1	9	31	10	10	17	23	5	5	0	33	12	5	5	0	44	1	0	0	50	0		
IV 2a	0	0	0	50	0	0	0	50	0	0	0	0	49	0	0	7	43	3	0	24	23	2	2	0	44	4	0	0	50	0	0			
I 1	0	0	0	50	4	3	1	2	44	3	0	3	2	45	4	2	11	35	22	15	7	18	10	6	5	1	41	3	3	3	0	42	5	
I 2	0	0	0	50	0	0	0	50	1	1	0	1	48	0	0	10	40	9	9	0	18	23	4	4	1	5	0	0	50	0	0	50	0	
III 1	0	0	0	50	0	0	0	50	0	0	2	48	8	0	1	4	11	8	3	12	2	7	0	0	0	49	1	1	0	47	2	0		
III 2	0	0	0	50	0	0	0	50	0	0	2	48	1	0	1	7	42	6	0	24	20	0	0	0	48	2	0	0	50	0	0	50	0	
IV 1	0	0	0	50	0	0	1	49	1	1	0	2	47	1	1	10	39	2	2	0	27	21	1	1	0	44	5	2	2	46	2	0		
IV 2	0	0	0	50	0	0	0	50	1	1	0	1	48	0	0	10	40	0	0	30	20	0	0	0	45	5	0	0	50	0	0	50	0	
Σ	2	2	0	448	9	5	4	11	430	27	20	7	32	391	35	31	4	100	315	67	53	14	212	171	21	19	2	402	27	7	6	1	431	12

TABLE 6.
FREQUENCIES OF THE DIFFERENT CLASSES OF JUDGMENTS. SUBJECT IV.

Series	84		88		92		96		100		104		108																						
	T	lg	h	l	T	lg	h	l	T	lg	h	l	T	lg	h	l																			
I 1	0	0	0	50	6	2	4	2	42	8	2	6	7	35	9	5	4	25	16	10	2	8	29	11	1	0	1	45	4	1	0	1	47	2	
I 2	1	0	1	49	2	2	0	48	5	3	2	1	44	11	4	7	14	25	7	4	3	29	12	3	3	0	45	2	4	3	1	43	3		
III 1	1	0	1	2	47	2	2	0	45	7	3	4	13	30	14	5	17	19	9	6	3	29	12	3	2	1	46	1	3	3	0	46	1		
III 2	1	0	1	1	48	5	3	2	3	42	7	3	4	8	35	11	7	4	19	20	6	2	4	37	7	5	2	3	42	3	5	4	1	45	0
IV 1	2	1	1	4	44	0	0	2	48	4	1	3	5	41	7	4	3	24	19	8	6	2	38	4	3	3	0	46	1	1	1	0	49	0	
IV 2	0	0	0	0	50	0	0	1	49	4	1	3	1	45	9	3	6	8	33	14	8	6	21	15	5	5	0	41	4	1	1	0	49	0	
Σ	5	1	4	7	288	15	9	6	8	277	35	13	22	35	230	61	28	33	107	132	54	28	26	183	63	20	15	5	965	15	15	12	3	279	6

TABLE 7.
FREQUENCIES OF THE DIFFERENT CLASSES OF JUDGMENTS. SUBJECT V.

Series	84			88			92			96			100			104			108																
	T	hg	lg	h	l	T	hg	lg	h	l	T	hg	lg	h	l	T	hg	lg	h	l	T	hg	lg	h	l										
I 1	0	0	0	2	48	4	3	1	0	46	6	3	3	8	36	13	13	0	18	19	17	16	1	26	7	17	17	0	29	4	9	8	1	41	0
I 2	0	0	0	0	50	7	7	0	1	42	9	6	3	6	35	15	11	4	16	19	19	15	4	20	11	7	7	0	41	2	1	1	0	48	1
III 2	1	0	1	0	49	0	0	0	0	50	4	3	1	9	37	17	16	1	21	12	0	0	0	39	11	2	1	1	46	2	1	1	0	48	1
III 1	0	0	0	1	49	0	0	0	2	48	1	1	0	8	41	0	0	0	23	27	1	1	0	29	20	2	2	0	47	1	0	0	0	48	2
IV 1	3	1	2	1	46	2	0	2	3	45	17	8	9	14	19	8	6	2	37	5	5	4	1	42	3	1	1	0	48	1	0	0	0	49	1
IV 2	0	0	0	4	46	2	1	1	1	47	5	2	3	3	42	6	3	3	39	5	1	0	1	48	1	0	0	0	50	0	0	0	0	49	1
Σ	4	1	3	8	288	15	11	4	7	278	42	23	19	48	210	59	49	10	154	87	43	36	7	204	53	29	28	1	261	10	11	10	1	283	6

TABLE 8.
FREQUENCIES OF THE DIFFERENT CLASSES OF JUDGMENTS. SUBJECT VI.

Series	84			88			92			96			100			104			108																
	T	hg	lg	h	l	T	hg	lg	h	l	T	hg	lg	h	l	T	hg	lg	h	l	T	hg	lg	h	l										
I 1	1	1	0	0	49	8	3	5	0	42	11	7	4	7	32	11	6	5	13	26	13	8	5	25	12	9	6	3	37	4	8	6	2	39	3
I 2	5	2	3	2	43	5	3	2	6	39	14	7	7	6	30	10	5	5	18	22	15	9	6	26	9	5	5	0	45	0	0	0	0	49	1
III 1	0	0	0	0	50	8	2	6	0	42	4	4	0	3	43	21	8	13	3	26	17	9	8	26	7	6	5	1	42	2	5	4	1	45	0
III 2	0	0	0	0	50	5	1	4	0	45	10	3	7	1	39	21	8	13	5	24	24	14	10	17	9	9	7	2	39	2	2	1	1	47	1
IV 1	0	0	0	0	50	6	2	4	1	43	15	12	3	2	33	14	5	9	6	30	22	11	11	15	13	12	7	5	33	5	9	5	4	40	1
IV 2	0	0	0	0	50	2	1	1	0	48	12	9	3	0	38	20	12	8	11	19	21	14	7	17	12	16	13	3	27	7	5	4	1	45	0
Σ	6	3	3	2	292	34	12	22	7	259	66	42	24	19	215	97	44	53	56	147	112	65	47	126	62	57	43	14	223	20	29	20	9	265	6

TABLE 11.
RELATIVE FREQUENCIES OF "H" AND OF "L" JUDGMENTS. SUBJECT II.

Series	84		88		92		96		100		104		108	
	h	l	h	l	h	l	h	l	h	l	h	l	h	l
I 2 a	0.020	0.920	0.020	0.900	0.080	0.740	0.220	0.580	0.440	0.220	0.620	0.240	0.940	0.000
IV 1 a	0.060	0.940	0.000	0.860	0.120	0.720	0.260	0.480	0.440	0.220	0.700	0.140	0.840	0.040
IV 2 a	0.020	0.980	0.000	0.880	0.040	0.820	0.300	0.460	0.580	0.240	0.860	0.120	0.900	0.000
I 1	0.040	0.960	0.040	0.900	0.200	0.660	0.340	0.460	0.480	0.260	0.760	0.180	0.940	0.000
I 2	0.000	0.980	0.020	0.820	0.160	0.680	0.340	0.460	0.500	0.240	0.780	0.120	0.840	0.040
III 1	0.000	0.960	0.020	0.860	0.060	0.700	0.300	0.420	0.420	0.400	0.940	0.020	0.920	0.060
III 2	0.020	0.920	0.060	0.820	0.080	0.720	0.320	0.380	0.540	0.180	0.960	0.000	0.900	0.000
IV 1	0.020	0.900	0.020	0.840	0.060	0.660	0.220	0.360	0.680	0.100	0.860	0.000	0.920	0.000
IV 2	0.020	0.940	0.040	0.880	0.200	0.600	0.340	0.440	0.680	0.220	0.860	0.040	0.940	0.000
Average	0.0222	0.9333	0.0244	0.9622	0.1111	0.7000	0.2633	0.4489	0.5289	0.2311	0.8156	0.0956	0.9044	0.0156

TABLE 12.
RELATIVE FREQUENCIES OF "H" AND OF "L" JUDGMENTS. SUBJECT III.

Series	84		88		92		96		100		104		108	
	h	l	h	l	h	l	h	l	h	l	h	l	h	l
I 2 a	0.000	1.000	0.060	0.900	0.240	0.540	0.240	0.240	0.520	0.300	0.920	0.020	0.920	0.060
IV 1 a	0.000	0.960	0.100	0.840	0.180	0.620	0.340	0.460	0.660	0.240	0.880	0.020	1.000	0.000
IV 2 a	0.000	1.000	0.000	1.000	0.020	0.980	0.140	0.860	0.480	0.460	0.880	0.080	1.000	0.000
I 1	0.000	1.000	0.040	0.880	0.040	0.900	0.220	0.700	0.360	0.200	0.820	0.060	0.840	0.100
I 2	0.000	1.000	0.000	1.000	0.020	0.960	0.200	0.800	0.360	0.460	0.820	0.100	1.000	0.000
III 1	0.000	1.000	0.000	1.000	0.040	0.960	0.020	0.820	0.240	0.540	0.980	0.020	0.940	0.040
III 2	0.000	1.000	0.000	1.000	0.040	0.940	0.140	0.840	0.480	0.400	0.960	0.040	1.000	0.000
IV 1	0.000	1.000	0.020	0.980	0.040	0.940	0.200	0.780	0.540	0.420	0.880	0.100	0.920	0.040
IV 2	0.000	1.000	0.000	1.000	0.020	0.960	0.200	0.800	0.600	0.400	0.900	0.100	1.000	0.000
Average	0.0000	0.6599	0.0244	0.9556	0.0711	0.8689	0.2222	0.7000	0.4711	0.3400	0.8933	0.0600	0.9578	0.0267

TABLE 13.
RELATIVE FREQUENCIES OF "H" AND OF "L" JUDGMENTS. SUBJECT IV.

Series	84		88		92		96		100		104		108	
	h	l	h	l	h	l	h	l	h	l	h	l	h	l
I 1	0.000	1.000	0.040	0.840	0.140	0.700	0.500	0.320	0.580	0.220	0.900	0.080	0.940	0.040
I 2	0.000	0.980	0.000	0.960	0.020	0.880	0.280	0.500	0.580	0.280	0.900	0.040	0.860	0.060
III 1	0.040	0.940	0.000	0.960	0.260	0.600	0.340	0.380	0.580	0.240	0.920	0.020	0.920	0.020
III 2	0.020	0.960	0.060	0.840	0.160	0.700	0.380	0.400	0.7 - 0	0.140	0.940	0.060	0.900	0.000
IV 1	0.080	0.880	0.040	0.960	0.100	0.820	0.480	0.380	0.760	0.080	0.920	0.020	0.980	0.000
IV 2	0.000	1.000	0.020	0.980	0.020	0.900	0.160	0.660	0.420	0.300	0.820	0.080	0.980	0.000
Average	0.0233	0.9600	0.0267	0.9233	0.1167	0.7667	0.3567	0.4400	0.6100	0.2100	0.8833	0.0500	0.9300	0.0200

TABLE 14.
RELATIVE FREQUENCIES OF "H" AND OF "L" JUDGMENTS. SUBJECT V.

Series	84		88		92		96		100		104		108	
	h	l	h	l	h	l	h	l	h	l	h	l	h	l
I 1	0.040	0.960	0.000	0.920	0.160	0.720	0.360	0.380	0.520	0.140	0.580	0.080	0.520	0.000
I 2	0.000	1.000	0.020	0.840	0.120	0.700	0.320	0.380	0.400	0.220	0.820	0.040	0.820	0.020
III 1	0.000	0.980	0.000	1.000	0.180	0.740	0.420	0.240	0.780	0.220	0.920	0.040	0.960	0.020
III 2	0.020	0.980	0.040	0.960	0.160	0.820	0.460	0.540	0.580	0.400	0.940	0.020	0.960	0.040
IV 1	0.020	0.920	0.030	0.900	0.280	0.380	0.740	0.100	0.840	0.060	0.960	0.020	0.980	0.020
IV 2	0.080	0.920	0.020	0.940	0.060	0.840	0.780	0.100	0.960	0.020	1.000	0.000	0.980	0.020
Average	0.0267	0.960	0.0233	0.9267	0.160	0.700	0.5133	0.290	0.680	0.1767	0.870	0.0333	0.9433	0.020

TABLE 15.
RELATIVE FREQUENCIES OF "H" AND OF "L" JUDGMENTS. SUBJECT VI.

Series	84		88		92		96		100		104		108	
	h	l	h	l	h	l	h	l	h	l	h	l	h	l
I 1	0.000	0.980	0.000	0.840	0.140	0.640	0.260	0.520	0.500	0.240	0.740	0.080	0.780	0.060
I 2	0.040	0.960	0.120	0.780	0.120	0.600	0.360	0.440	0.520	0.180	0.900	0.000	0.980	0.020
III 1	0.000	1.000	0.000	0.840	0.060	0.860	0.060	0.520	0.520	0.140	0.840	0.040	0.900	0.000
III 2	0.000	1.000	0.000	0.900	0.020	0.780	0.100	0.480	0.340	0.180	0.780	0.040	0.940	0.020
IV 1	0.000	1.000	0.020	0.860	0.040	0.660	0.120	0.600	0.300	0.260	0.660	0.100	0.800	0.020
IV 2	0.000	1.000	0.000	0.960	0.000	0.760	0.220	0.380	0.340	0.240	0.540	0.140	0.900	0.000
Average	0.0067	0.9733	0.0233	0.8633	0.0633	0.7167	0.1867	0.4900	0.4200	0.2067	0.7433	0.0667	0.8833	0.0200

TABLE 16.
RELATIVE FREQUENCIES OF "H" AND OF "L" JUDGMENTS. SUBJECT VII.

Series	84		88		92		96		100		104		108	
	h	l	h	l	h	l	h	l	h	l	h	l	h	l
I 1	0.020	0.980	0.020	0.880	0.140	0.680	0.160	0.400	0.440	0.260	0.700	0.040	0.640	0.040
I 2	0.000	0.960	0.000	0.920	0.020	0.720	0.020	0.520	0.360	0.300	0.600	0.160	0.840	0.000
III 1	0.000	0.980	0.000	0.900	0.060	0.780	0.260	0.460	0.400	0.280	0.820	0.080	0.800	0.080
III 2	0.020	0.980	0.020	0.860	0.020	0.880	0.120	0.580	0.300	0.440	0.760	0.040	0.800	0.080
IV 1	0.000	1.000	0.000	0.940	0.040	0.800	0.200	0.360	0.480	0.140	0.800	0.040	0.900	0.020
IV 2	0.000	0.960	0.000	0.880	0.120	0.580	0.260	0.420	0.620	0.180	0.860	0.000	0.940	0.020
Average	0.0067	0.9800	0.0067	0.8967	0.0667	0.7400	0.1700	0.4567	0.4367	0.2667	0.7567	0.0600	0.8200	0.0400

TABLE 17.

m	n	p'	q'	P. E.
0	50	0.01923	0.98077	0.00000
1	49	.03846	.96154	.01335
2	48	.05769	.94231	.01869
3	47	.07692	.92308	.02265
4	46	.09615	.90385	.02588
5	45	.11538	.88462	.02862
6	44	.13462	.86538	.03100
7	43	.15385	.84615	.03310
8	42	.17308	.82692	.03497
9	41	.19231	.80769	.03665
10	40	.21154	.78846	.03816
11	39	.23077	.76923	.03951
12	38	.25000	.75000	.04074
13	37	.26923	.73077	.04184
14	36	.28846	.71154	.04283
15	35	.30769	.69231	.04371
16	34	.32692	.67308	.04450
17	33	.34614	.65386	.04519
18	32	.36538	.63462	.04579
19	31	.38462	.61538	.04630
20	30	.40385	.59615	.04673
21	29	.42308	.57692	.04708
22	28	.44231	.55769	.04735
23	27	.46154	.53846	.04754
24	26	.48077	.51923	.04766
25	25	.50000	.50000	.04769

TABLE 18.

COEFFICIENTS OF DIVERGENCE. SUBJECT I.

	h'	h	Q
84 h	106.72	106.06	1.00
84 l	20.37	32.35	0.63
88 h	35.71	28.87	1.24
88 l	14.25	7.19	1.98
92 h	17.57	13.13	1.34
92 l	10.13	4.23	2.39
96 h	12.03	5.77	2.08
96 l	10.59	5.99	1.77
100 h	10.15	4.28	2.37
100 l	15.27	13.25	1.15
104 h	16.35	16.01	1.02
104 l	43.65	50.00	0.87
108 h	21.05	19.61	1.07
108 l	61.29	89.44	0.68

TABLE 19.
COEFFICIENTS OF DIVERGENCE. SUBJECT II.

	h'	h	Q
84 h	33.94	40.00	0.85
84 l	20.02	18.26	1.10
88 h	32.41	36.51	0.89
88 l	14.51	23.10	0.63
92 h	15.91	11.45	1.39
92 l	10.91	11.47	0.95
96 h	10.98	14.43	0.76
96 l	10.05	11.16	0.90
100 h	10.01	7.10	1.41
100 l	11.86	8.98	1.32
104 h	12.89	6.14	2.14
104 l	11.86	8.29	1.43
108 h	17.00	17.82	0.95
108 l	40.35	29.49	1.37

TABLE 20.
COEFFICIENTS OF DIVERGENCE. SUBJECT III.

	h'	h	Q
84 h			
84 l	75.54	53.45	1.41
88 h	32.41	19.80	1.64
88 l	24.27	11.23	2.16
92 h	19.45	8.75	2.22
92 l	14.81	4.26	3.48
96 h	12.03	4.84	2.48
96 l	10.91	3.35	3.26
100 h	10.02	5.39	1.86
100 l	10.30	6.36	1.62
104 h	16.20	12.91	1.25
104 l	21.05	19.61	1.07
108 h	24.87	12.35	2.01
108 l	31.02	19.61	1.58

TABLE 21.
COEFFICIENTS OF DIVERGENCE. SUBJECT IV.

	h'	h	Q
84 h	33.14	22.14	1.50
84 l	25.51	15.50	1.65
88 h	31.02	29.36	1.06
88 l	18.79	10.89	1.72
92 h	15.57	8.73	1.78
92 l	11.82	5.98	1.98
96 h	10.44	6.12	1.71
96 l	10.07	5.77	1.75
100 h	10.25	5.01	2.05
100 l	12.28	8.36	1.47
104 h	15.57	16.57	0.94
104 l	22.94	25.65	0.89
108 h	19.60	15.08	1.30
108 l	35.72	27.95	1.28

TABLE 22.
COEFFICIENTS OF DIVERGENCE. SUBJECT V.

	h'	h	Q
84 h	31.02	23.57	1.31
84 l	25.51	21.13	1.20
88 h	33.15	30.43	1.09
88 l	19.18	12.95	1.48
92 h	13.64	9.73	1.40
92 l	10.91	4.25	2.56
96 h	10.00	3.78	2.64
96 l	10.45	4.04	2.58
100 h	10.72	3.31	3.23
100 l	13.11	5.18	2.53
104 h	14.87	4.58	3.24
104 l	27.87	25.65	1.08
108 h	21.62	18.70	1.15
108 l	35.71	79.05	0.45

TABLE 23.
COEFFICIENTS OF DIVERGENCE. SUBJECT VI.

	h'	h	Q
84 h	61.29	43.85	1.39
84 l	26.45	12.62	2.09
88 h	33.14	14.74	2.25
88 l	14.55	11.56	1.26
92 h	20.53	12.70	1.62
92 l	11.10	7.14	1.56
96 h	12.83	6.22	2.06
96 l	10.00	9.35	1.07
100 h	10.13	6.83	1.48
100 l	12.35	15.14	0.81
104 h	11.44	5.43	2.09
104 l	20.04	14.14	1.42
108 h	15.57	9.02	1.72
108 l	35.72	32.27	1.10

TABLE 24.
COEFFICIENTS OF DIVERGENCE. SUBJECT VII.

	h'	h	Q
84 h	61.29	111.8	0.55
84 l	35.71	79.06	0.45
88 h	61.29	70.71	0.87
88 l	16.43	24.69	0.66
92 h	20.04	13.71	1.46
92 l	11.40	6.78	1.68
96 h	13.29	7.70	1.73
96 l	10.04	8.69	1.15
100 h	10.08	6.52	1.54
100 l	11.31	6.76	1.67
104 h	11.65	7.51	1.55
104 l	21.05	1.83	1.64
108 h	13.01	6.78	1.92
108 l	25.52	21.13	1.20

TABLE 25.
RELATIVE FREQUENCY OF "HEAVIER" JUDGMENTS IN THE DIFFERENT COLUMNS.

Subject	1	2	3	4	5
I	0.5619	0.5488	0.5250	0.5464	0.5048
II	0.4824	0.5004	0.5333	0.4929	0.5000
III	0.4345	0.4119	0.4238	0.4095	0.4119
IV	0.5554	0.5143	0.5411	0.5321	0.5607
V	0.5929	0.5911	0.5643	0.5821	0.5696
VI	0.4696	0.4964	0.5250	0.4643	0.4536
VII	0.4589	0.4554	0.4411	0.5018	0.4732

TABLE 26.

Subject	s	$10^8 \sum (\frac{m_i}{s} - p)^2$	p	u	q
I	840	202773	0.5374	0.0101	0.4626
II	840	146678	0.5017	0.0085	0.4983
III	840	45255	0.4183	0.0071	0.5817
IV	560	138717	0.5407	0.0083	0.4593
V	560	64868	0.5800	0.0057	0.4200
VI	560	332973	0.4818	0.0129	0.5182
VII	560	211623	0.4661	0.0103	0.5339

TABLE 27.
COEFFICIENT OF DIVERGENCE AND COMPONENT OF PHYSICAL VARIATION.

Subject	Q	M
I	1.462	0.020126
II	1.408	0.017119
III	1.145	0.009507
IV	1.290	0.016641
V	1.139	0.011380
VI	1.578	0.025782
VII	1.396	0.020554

AVERAGE OF Q = 1.345

TABLE 28.—RELATIVE FREQUENCY OF "H" AND "NOT-H" JUDGMENTS.

	I		II		III		IV		V		VI		VII	
	h	not-h	h	not-h	h	not-h	h	not-h	h	not-h	h	not-h	h	not-h
84	0.0022	0.9978	0.0222	0.9778	0.0000	1.0000	0.0233	0.9767	0.0267	0.9733	0.0067	0.9933	0.0067	0.9933
88	0.0200	0.9800	0.0244	0.9756	0.0244	0.9756	0.0267	0.9733	0.0233	0.9767	0.0233	0.9767	0.0067	0.9933
92	0.0889	0.9111	0.1111	0.8889	0.0711	0.9289	0.1167	0.8833	0.1600	0.8400	0.0633	0.9367	0.0667	0.9333
96	0.2222	0.7778	0.2933	0.7067	0.2222	0.7778	0.3567	0.6433	0.5133	0.4867	0.1867	0.8133	0.1700	0.8300
100	0.4133	0.5867	0.5289	0.4711	0.4711	0.5289	0.6100	0.3900	0.6800	0.3200	0.4200	0.5800	0.4367	0.5633
104	0.8956	0.1044	0.8156	0.1844	0.8933	0.1067	0.8833	0.1167	0.8700	0.1300	0.7433	0.2567	0.7567	0.2433
108	0.9400	0.0600	0.9044	0.0956	0.9578	0.0422	0.9300	0.0700	0.9433	0.0567	0.8833	0.1167	0.8200	0.1800

TABLE 29.

VALUES OF THE P'S FOR THE DETERMINATION OF THE JUST PERCEPTIBLE POSITIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
84	0.0022	0.0222	0.0000	0.0233	0.0267	0.0067	0.0067
88	0.0200	0.0238	0.0244	0.0260	0.0227	0.0232	0.0066
92	0.0869	0.1060	0.0694	0.1109	0.1521	0.0614	0.0658
96	0.1980	0.2487	0.2014	0.2995	0.4099	0.1696	0.1566
100	0.2864	0.3169	0.3320	0.3295	0.2842	0.3104	0.3337
104	0.3641	0.2302	0.3330	0.1861	0.1082	0.3186	0.3258
108	0.0399	0.0471	0.0381	0.0229	0.0153	0.0972	0.0859
Σ	0.9975	0.9949	0.9983	0.9982	0.9991	0.9871	0.9811
R	0.0025	0.0050	0.0017	0.0017	0.0009	0.0128	0.0189

TABLE 30.
VALUES OF $I_K P_K$ FOR THE DETERMINATION OF THE JUST PERCEPTIBLE POSITIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
84	0.1848	1.8648	0.0000	1.9572	2.2428	0.5628	0.5628
88	1.7600	2.0944	2.1474	2.2880	1.9976	2.0416	0.5824
92	9.9948	9.7520	6.3848	10.2028	13.9832	5.6488	6.0536
96	19.0080	23.8752	19.3344	28.7520	39.3504	16.2816	15.0336
100	28.6410	31.6910	33.2030	32.9515	26.4247	31.0414	33.3662
104	37.8964	23.9408	34.6320	19.3544	11.2528	33.1344	33.8832
108	4.3092	5.0868	4.1148	2.4732	1.6524	10.4976	9.2772
Σ	99.7642	98.3050	99.8164	97.9791	96.9139	99.2082	98.7590

TABLE 31.
VALUES OF $I_K^2 P_K$ FOR THE DETERMINATION OF THE JUST PERCEPTIBLE POSITIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
84	15.5232	156.6432	00.0000	164.4048	188.3952	47.2752	47.2752
88	154.8900	184.3072	188.9712	201.3440	175.7888	179.6608	51.2512
92	919.5216	897.1840	587.4016	938.6576	1287.3744	519.6896	556.9312
96	1824.7680	2292.0192	1856.1024	2760.1920	3777.6384	1563.0336	1443.2256
100	2864.1000	3169.1000	3320.3000	3295.1500	2642.4700	3104.1400	3336.6200
104	3938.1056	2489.8432	3601.7280	2012.8576	1170.2912	3445.9776	3523.8528
108	465.3936	549.3744	444.3984	287.1056	178.4592	1133.7408	1001.0376
Σ	10182.2920	9738.4712	9998.9016	9638.7116	9420.4172	9993.5176	9961.0936

TABLE 32.

OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT I.

r_k	IVa		I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	—	—	—	—	—	—	—	—
88	—	—	3	264	1	88	3	264
92	9	828	8	736	7	644	6	552
96	16	1536	24	2304	20	1920	16	1536
100	15	1500	15	1500	36	3600	40	4000
104	58	6032	43	4472	35	3640	33	3432
108	2	216	7	756	1	108	2	216
Σ	100	10112	100	10032	100	10000	100	10000
Average		101.12		100.32		100.00		100.00

TABLE 33.

OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT II.

r_k	IVa		I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	4	336	2	168	1	84	1	84
88	—	—	3	264	4	352	3	264
92	8	736	18	1656	7	644	12	1104
96	25	2400	26	2496	27	2592	23	2208
100	30	3000	23	2300	28	2800	37	3700
104	24	2496	19	1976	31	3224	21	2184
108	8	864	6	648	2	216	2	216
Σ	99	9832	97	9508	100	9912	99	9760
Average		99.31		98.02		99.12		98.59

TABLE 34.

OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT III.

r_k	IVa		I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	—	—	1	84	—	—	—	—
88	4	352	3	264	—	—	1	88
92	10	920	19	1748	4	368	3	276
96	21	2016	36	3456	8	768	20	1920
100	36	3600	34	3400	32	3200	41	4100
104	28	2912	5	520	56	5824	30	3120
108	1	108	2	216	—	—	5	540
Σ	100	9908	100	9688	100	10160	100	10044
Average		99.08		96.88		101.60		100.44

TABLE 35.

OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT IV.

r_k	I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	—	—	2	168	4	336
88	2	176	3	264	1	88
92	8	736	20	1840	4	368
96	35	3360	28	2688	27	2592
100	32	3200	24	2400	34	3400
104	22	2288	20	2080	25	2600
108	1	108	3	324	5	540
Σ	100	9868	100	9764	100	9924
Average		98.68		97.64		99.24

TABLE 36.

OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT V.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	2	168	1	84	—	—
88	1	88	—	—	3	264
92	9	828	16	1472	13	1196
96	29	2784	36	3456	63	6048
100	30	3000	35	3500	17	1700
104	19	1976	12	1248	4	416
108	10	1080	—	—	—	—
Σ	100	9924	100	9760	100	9624
Average		99.24		97.60		96.24

TABLE 37.

OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT VI.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	2	168	—	—	—	—
88	6	528	—	—	1	88
92	13	1196	5	460	2	184
96	25	2400	8	768	18	1728
100	31	3100	44	4400	20	2000
104	21	2184	37	3848	32	3328
108	1	108	6	648	23	2484
Σ	99	9684	100	10124	96	9812
Average		97.82		101.24		102.21

TABLE 38.

OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT VII.

r_k	I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	1	84	1	84	—	—
88	1	88	1	88	—	—
92	7	644	4	368	8	736
96	6	576	18	1728	19	1824
100	35	3500	27	2700	39	3900
104	31	3224	38	3952	30	3120
108	19	2052	7	756	4	432
Σ	100	10168	96	9676	100	10012
Average		101.68		100.79		100.12

TABLE 39.

DIFFERENCES BETWEEN OBSERVED AND CALCULATED VALUES OF THE JUST PERCEPTIBLE POSITIVE DIFFERENCE.

Subject	Differences between observed and calculated values.					
	Calculated value	IVa	I	III	IV	Total
I	99.76	+1.36	+0.56	+0.24	+0.24	+0.60
II	98.31	+1.01	-0.29	+0.81	+0.28	+0.40
III	99.82	-0.74	-0.94	+1.78	+0.62	-0.32
IV	97.98		+0.70	-0.34	+1.26	+0.54
V	96.91		+2.33	+0.69	-0.67	+0.78
VI	99.21		-1.39	+2.03	+3.00	+1.20
VII	98.76		+2.92	+2.03	+1.36	+2.17

TABLE 40.
OBSERVATIONS ON THE JUST PERCEPTIBLE POSITIVE DIFFERENCE. COMBINED RESULTS.

Γ_k	I		II		III		IV		V		VI		VII	
	N_k	$\Gamma_k N_k$	N_k	$\Gamma_k N_k$	N_k	$\Gamma_k N_k$	N_k	$\Gamma_k N_k$	N_k	$\Gamma_k N_k$	N_k	$\Gamma_k N_k$	N_k	$\Gamma_k N_k$
84	-	-	8	672	1	84	6	504	3	252	2	168	2	168
88	7	616	10	880	8	704	6	528	4	352	7	616	2	176
92	30	2760	45	4140	36	3312	32	2944	38	3496	20	1840	19	1748
96	76	7296	101	9696	85	8160	90	8640	128	12288	51	4896	43	4128
100	108	10800	118	11800	143	14300	90	9000	82	8200	95	9500	101	10100
104	169	17576	95	9880	119	12376	67	6988	35	3640	90	9360	99	10296
108	12	1296	18	1944	8	864	9	972	10	1080	30	3240	30	3240
Σ	400	40144	395	39012	400	39800	300	29556	300	29308	295	29620	296	29856
Average		100.360		98.714		99.500		98.520		97.693		100.407		100.856

TABLE 41.
VALUES OF THE P'S FOR THE DETERMINATION OF THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
108	0.0600	0.0956	0.0422	0.0700	0.0567	0.1167	0.1800
104	0.0981	0.1668	0.1022	0.1085	0.1226	0.2267	0.1995
100	0.4939	0.3475	0.4525	0.3204	0.2626	0.3808	0.3495
96	0.2706	0.2757	0.3135	0.3224	0.2716	0.2243	0.2249
92	0.0704	0.1017	0.0832	0.1679	0.2406	0.0482	0.0430
88	0.0070	0.0134	0.0062	0.0203	0.0448	0.0032	0.0031
84	0.0000	0.0003	0.0002	0.0005	0.0010	0.0001	0.0000
Σ	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000
R	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 42.
VALUES OF $I_K P_K$ FOR THE DETERMINATION OF THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
108	6.4800	10.3248	4.5576	7.5600	6.1236	12.6036	19.4400
104	10.2024	17.3472	10.6288	11.2840	12.7504	23.5768	20.7480
100	49.3860	34.7520	45.2530	32.0370	26.2610	38.0800	34.9520
96	25.9776	26.4672	30.0960	30.9504	26.0736	21.5328	21.5904
92	6.4768	9.3564	7.6544	14.5268	22.1352	4.4344	3.9560
88	0.6160	1.0912	0.5456	1.7864	3.9424	0.2816	0.2728
84	0.0000	0.0252	0.0168	0.0420	0.0840	0.0084	0.0000
Σ	99.1388	99.3640	98.7522	98.1866	97.3702	100.5176	100.9592

TABLE 43.
VALUES OF $I_K^2 P_K$ FOR THE DETERMINATION OF THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE.

I_K	I	II	III	IV	V	VI	VII
108	699.8400	1115.0784	492.2208	816.4800	661.3488	1361.1888	2099.5200
104	1061.0496	1804.1088	1105.3952	1173.5360	1326.0416	2451.9872	2157.7920
100	4938.6000	3475.2000	4525.3000	3203.7000	2626.1000	3808.0000	3495.2000
96	2493.8496	2540.8512	2889.2160	2971.2384	2503.0656	2067.1488	2072.6784
92	595.8656	860.7888	704.2048	1336.4656	2036.4384	407.9648	363.9520
88	54.2080	96.0256	48.0128	187.2032	346.9312	24.7808	24.0064
84	0.0000	2.1168	1.4112	3.5280	7.0560	0.7056	0.0000
Σ	9843.4128	9894.1696	9765.7608	9662.1512	9506.9816	10121.7760	10213.1488

TABLE 44.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT I.

r_k	IVa		I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$N_k r_k$	N_k	$r_k N_k$
84								
88	5	440	2	176			1	88
92	9	828	12	1104	9	828	6	552
96	15	1440	13	1248	31	2976	40	3840
100	58	5800	56	5600	44	4400	42	4200
104	5	520	11	1144	11	1144	8	832
108	8	864	6	648	5	540	3	324
Σ	100	9892	100	9920	100	9888	100	9836
Average		98.92		99.20		98.88		98.39

TABLE 45.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT II.

r_k	IVa		I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84							1	84
88	1	88	4	352	2	176	3	264
92	13	1196	12	1104	11	1012	14	1288
96	27	2592	20	1920	33	3168	38	3648
100	28	2800	35	3500	45	4500	25	2500
104	17	1768	19	1976	5	520	12	1248
108	14	1512	10	1080	4	432	7	756
Σ	100	9956	100	9932	100	9808	100	9788
Average		99.56		99.32		98.08		97.88

TABLE 46.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT III.

	IVa		I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84								
88	2	176					1	88
92	10	920	2	184	4	368	5	460
96	33	3168	29	2784	17	1632	18	1728
100	45	4500	48	4800	43	4300	40	4000
104	10	1040	14	1456	36	3744	31	3224
108			7	756			5	540
Σ	100	9804	100	9980	100	10044	100	10040
Average		98.04		99.80		100.44		100.40

TABLE 47.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT IV.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$N_k k$
84			1	84		
88	2	176	5	440	2	176
92	18	1656	19	1748	18	1656
96	27	2592	25	240	28	2688
100	38	3800	29	2900	36	3600
104	8	832	12	1248	13	1352
108	7	756	9	972	2	216
Σ	100	9812	100	9792	99	9688
Average		98.12		97.92		97.86

TABLE 48.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT V.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84						
88			4	352	8	704
92	8	736	19	1748	59	5428
96	24	2304	39	3744	20	1920
100	30	3000	28	2800	10	1000
104	28	2912	6	624	2	208
108	10	1080	4	432	1	108
Σ	100	10032	100	9700	100	9368
Average		100.32		97.00		93.68

TABLE 49.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT VI.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84						
88	2	176				
92	12	1104	2	184	4	368
96	25	2400	32	3072	12	1152
100	36	3600	44	4400	33	3300
104	14	1456	16	1664	36	3744
108	11	1188	6	648	15	1620
Σ	100	9924	100	9968	100	10184
Average		99.24		99.68		101.84

TABLE 50.
OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. SUBJECT VII.

r_k	I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	1	84	1	84		
88	4	352	4	352	1	88
92	14	1288	9	828	16	1472
96	21	2016	37	3552	24	2304
100	33	3300	28	2800	42	4200
104	22	2288	17	1768	16	1664
108	5	540	3	324	1	108
Σ	100	9868	99	9708	100	9836
Average		98.68		98.06		98.36

TABLE 51.
OBSERVED VALUES OF JUST PERCEPTIBLE AND JUST IMPERCEPTIBLE POSITIVE DIFFERENCE.

Subject.	IVa		I		III		IV	
	A	B	A	B	A	B	A	B
I	101.12	98.92	100.32	99.20	100.00	98.88	100.00	98.36
II	99.47	99.56	98.02	99.32	99.12	98.08	98.59	97.88
III	99.08	98.04	96.88	99.80	101.60	100.44	100.44	100.40
IV			98.68	98.12	97.64	97.92	99.24	97.86
V			99.24	100.32	97.60	97.00	96.24	98.68
IV			97.82	99.24	101.24	99.68	102.21	101.84
VII			101.68	98.68	100.79	98.06	100.12	98.36

TABLE 52.
OBSERVATIONS ON THE JUST IMPERCEPTIBLE POSITIVE DIFFERENCE. COMBINED RESULTS.

r_k	I		II		III		IV		V		VI		VII	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	8	704	1	84	3	264	1	84	12	1056	2	176	2	168
88	36	3312	10	880	21	1932	9	792	86	7912	18	1656	9	792
92	99	9504	50	4600	97	9312	55	5060	83	7988	69	6624	39	3588
96	200	20000	118	11328	133	13300	80	7680	88	7988	82	7872	82	7872
100	35	3640	53	5512	91	9464	103	10300	68	6800	113	11300	103	10300
104	22	2276	35	3780	12	1296	33	3432	36	3744	66	6864	55	5720
108							18	1944	15	1620	32	3456	9	972
Σ	400	39536	400	39484	400	39868	299	29292	300	29100	300	30076	299	29412
Average		98.840		98.710		99.670		97.967		97.000		100.253		98.368

TABLE 53.
RELATIVE FREQUENCY OF "1" AND "not-1" JUDGMENTS.

r_k	I		II		III		IV		V		VI		VII	
	1	not-1	1	not-1	1	not-1	1	not-1	1	not-1	1	not-1	1	not-1
84	0.9356	0.0644	0.9333	0.0667	0.9956	0.0044	0.9600	0.0400	0.9600	0.0400	0.9733	0.0267	0.9800	0.0200
88	0.8556	0.1444	0.8622	0.1378	0.9556	0.0444	0.9233	0.0767	0.9267	0.0733	0.8633	0.1367	0.8967	0.1033
92	0.5800	0.4200	0.7000	0.3000	0.8889	0.1111	0.7667	0.2333	0.7000	0.3000	0.7167	0.2833	0.7400	0.2600
96	0.3356	0.6644	0.4489	0.5511	0.7000	0.3000	0.4400	0.5600	0.2900	0.7100	0.4900	0.5100	0.4567	0.5433
100	0.1222	0.8778	0.2311	0.7689	0.3800	0.6200	0.2100	0.7900	0.1767	0.8233	0.2067	0.7933	0.2667	0.7333
104	0.0133	0.9867	0.0956	0.9044	0.0600	0.9400	0.0500	0.9500	0.0333	0.9667	0.0667	0.9333	0.0600	0.9400
108	0.0067	0.9933	0.0156	0.9844	0.0267	0.9733	0.0200	0.9800	0.0200	0.9800	0.0200	0.9800	0.0400	0.9600

TABLE 54.

VALUES OF THE U 'S FOR THE DETERMINATION OF THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
108	0.0067	0.0156	0.0267	0.0200	0.0200	0.0200	0.0400
104	0.0132	0.0941	0.0584	0.0490	0.0327	0.0653	0.0576
100	0.1198	0.2058	0.3477	0.1955	0.1674	0.1890	0.2406
96	0.2887	0.3073	0.3971	0.3236	0.2262	0.3556	0.3022
92	0.3315	0.2641	0.1479	0.3158	0.3876	0.2652	0.2661
88	0.2054	0.0976	0.0213	0.0887	0.1539	0.0905	0.0838
84	0.0324	0.0145	0.0009	0.0071	0.0117	0.0139	0.0095
Σ	0.9977	0.9990	1.0000	0.9997	0.9995	0.9995	0.9998
R	0.0022	0.0010	0.0000	0.0003	0.0005	0.0004	0.0002

TABLE 55.

VALUES OF r_k U_k FOR THE DETERMINATION OF THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
84	2.7216	1.3104	0.0756	0.5964	0.9828	1.1676	0.7980
88	18.0752	8.5888	1.8744	7.8056	13.5432	7.9640	7.3744
92	30.4980	24.2972	13.6068	28.9436	35.6592	24.3984	24.4812
96	27.7152	29.5008	38.1216	31.0756	21.7152	34.1376	29.0112
100	11.9785	20.5764	34.7685	19.5509	16.7354	18.9030	24.0627
104	1.3728	9.7864	6.0736	5.0960	3.4008	6.7912	5.9904
108	0.7236	1.2180	2.8836	2.1600	2.1600	2.1600	4.3200
Σ	93.0849	95.2780	97.4041	95.2281	94.1966	95.5218	96.0379

TABLE 56.
VALUES OF $r_k^2 U_k$ FOR THE DETERMINATION OF THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE.

	I	II	III	IV	V	VI	VII
84	228.6144	110.0736	6.3504	50.0976	82.5552	98.0784	67.0320
88	1590.6176	755.8144	164.9472	686.8928	1191.8016	700.8320	648.9472
92	2805.8160	2235.3424	1251.8256	2062.8112	3280.6464	2244.6528	2252.2704
96	2660.6592	2632.0768	3659.6736	2083.2576	2084.6592	3277.2096	2785.0752
100	1197.8500	2057.6400	3476.8500	1955.0900	1673.5400	1890.3000	2406.2700
104	142.7712	1017.7856	631.6544	529.9840	353.6832	706.2848	623.0016
108	78.1488	131.5440	311.4288	233.2800	233.2800	233.2800	466.5600
Σ	8704.4772	9140.2768	9502.7300	9101.4132	8900.1656	9150.6376	9249.1564

TABLE 57.
OBSERVATIONS ON THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT I.

r_k	IVa		I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	5	420	3	252	1	84	3	252
88	27	2376	38	3344	13	1144	11	968
92	42	3864	20	1840	33	3036	38	3496
96	20	1920	28	2688	33	3168	31	2976
100	6	600	10	1000	16	1600	11	1100
104			1	104	2	208	3	312
108					1	108	2	206
Σ	100	9180	100	9228	99	9348	99	9320
Average		91.80		92.28		94.43		94.14

TABLE 58.
OBSERVATIONS ON THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT II.

	IVa		I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	2	168	1	84	2	168	4	336
88	6	528	12	1056	13	1144	17	1496
92	30	2760	23	2116	28	2576	33	3036
96	29	2784	29	2784	24	2304	30	2880
100	21	2100	19	1900	28	2800	14	1400
104	10	1040	13	1352	1	104	2	208
108	2	216	3	324	4	432		
Σ	100	9596	100	9616	100	9528	100	9356
Average		95.96		96.16		95.28		93.56

TABLE 59.
OBSERVATIONS ON THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT III.

	IVa		I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	2	168						
88	4	352	2	176			1	88
92	17	1564	13	1196	9	828	13	1196
96	38	3648	45	4320	42	4032	40	3840
100	34	3400	27	2700	45	4500	34	3400
104	5	520	8	832	2	208	10	1040
108			5	540	2	216	2	216
Σ	100	9652	100	9764	100	9784	100	9780
Average		96.52		97.64		97.84		97.80

TABLE 60.

OBSERVATIONS ON THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT IV.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	1	84	1	84	2	168
88	15	1320	6	528	5	440
92	35	3220	36	3312	37	3404
96	28	2688	34	3264	22	2112
100	17	1700	17	1700	24	2400
104	3	312	5	520	6	624
108	1	108			4	432
Σ	100	9432	99	9408	100	9580
Average		94.32		95.03		95.80

TABLE 61.

OBSERVATIONS ON THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT V.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	1	84			3	252
88	10	880	8	704	27	2376
92	33	3036	30	2760	53	4876
96	31	2976	28	2688	10	960
100	18	1800	28	2800	4	400
104	6	624	3	312	1	104
108	1	108	3	324	1	108
Σ	100	9508	100	9588	99	9076
Average		95.08		95.88		91.68

TABLE 62.
OBSERVATIONS ON THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT VI.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	1	84	1	84		
88	16	1408	7	616	10	880
92	25	2300	32	2944	24	2208
96	32	3072	42	4032	32	3072
100	19	1900	13	1300	23	2300
104	4	416	4	416	11	1144
108	3	324	1	108		
Σ	100	9504	100	9500	100	9604
Average		95.04		95.00		96.04

TABLE 63.
OBSERVATIONS ON THE JUST PERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT VII.

	I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84					2	168
88	9	792	1	88	16	1408
92	28	2576	24	2208	27	2484
96	28	2688	31	2976	35	3360
100	23	2300	31	3100	16	1600
104	10	1040	5	520	2	208
108	2	216	8	864	2	216
Σ	100	9612	100	9756	100	9444
Average		96.12		97.56		94.44

TABLE 64

TABLE OF THEORETICAL DISTRIBUTION OF CORRELATION COEFFICIENTS

r	11		12		Y		VI		VII	
	f_{11}	f_{12}	f_{11}	f_{12}	f_{11}	f_{12}	f_{11}	f_{12}	f_{11}	f_{12}
2	0.0	0.0	4	0.0	4	0.00	7	1.08	2	108
3	0.0	0.0	0.0	2.0	4.0	0.00	0.0	2.04	26	2208
4	0.0	0.0	0.0	0.0	0.0	1.00	0.1	4.02	39	7208
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	10.16	94	9024
6	0.0	0.0	0.0	0.0	0.0	0.00	0.0	0.00	70	5000
7	0.0	0.0	0.0	0.0	0.0	0.00	0.0	19.16	17	1708
8	0.0	0.0	0.0	0.0	0.0	0.00	0.0	4.32	12	1200
9	0.0	0.0	0.0	2.0	2.0	2.0	0.0	28.08	300	28812
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Observations on		M		M		M	
1		1		1		1	
2		2		2		2	
3		3		3		3	
4		4		4		4	
5		5		5		5	
6		6		6		6	
7		7		7		7	
8		8		8		8	
9		9		9		9	
10		10		10		10	
11		11		11		11	
12		12		12		12	
13		13		13		13	
14		14		14		14	
15		15		15		15	
16		16		16		16	
17		17		17		17	
18		18		18		18	
19		19		19		19	
20		20		20		20	
21		21		21		21	
22		22		22		22	
23		23		23		23	
24		24		24		24	
25		25		25		25	
26		26		26		26	
27		27		27		27	
28		28		28		28	
29		29		29		29	
30		30		30		30	
31		31		31		31	
32		32		32		32	
33		33		33		33	
34		34		34		34	
35		35		35		35	
36		36		36		36	
37		37		37		37	
38		38		38		38	
39		39		39		39	
40		40		40		40	
41		41		41		41	
42		42		42		42	
43		43		43		43	
44		44		44		44	
45		45		45		45	
46		46		46		46	
47		47		47		47	
48		48		48		48	
49		49		49		49	
50		50		50		50	
51		51		51		51	
52		52		52		52	
53		53		53		53	
54		54		54		54	
55		55		55		55	
56		56		56		56	
57		57		57		57	
58		58		58		58	
59		59		59		59	
60		60		60		60	
61		61		61		61	
62		62		62		62	
63		63		63		63	
64		64		64		64	
65		65		65		65	
66		66		66		66	
67		67		67		67	
68		68		68		68	
69		69		69		69	
70		70		70		70	
71		71		71		71	
72		72		72		72	
73		73		73		73	
74		74		74		74	
75		75		75		75	
76		76		76		76	
77		77		77		77	
78		78		78		78	
79		79		79		79	
80		80		80		80	
81		81		81		81	
82		82		82		82	
83		83		83		83	
84		84		84		84	
85		85		85		85	
86		86		86		86	
87		87		87		87	
88		88		88		88	
89		89		89		89	
90		90		90		90	
91		91		91		91	
92		92		92		92	
93		93		93		93	
94		94		94		94	
95		95		95		95	
96		96		96		96	
97		97		97		97	
98		98		98		98	
99		99		99		99	
100		100		100		100	

TABLE 65.

DIFFERENCES BETWEEN OBSERVED AND CALCULATED VALUES OF THE JUST PERCEPTIBLE
NEGATIVE DIFFERENCE.

Subject	Differences between observed and calculated values.					
	Calculated value	IV _a	I	III	IV	Totals
I	93.08	-1.28	-0.80	+1.35	+1.06	+0.08
II	95.28	+0.68	+0.88	0.00	-1.72	-0.04
III	97.40	-0.88	+0.23	+0.44	+0.40	+0.05
IV	95.23		-0.91	-0.20	+0.57	-0.18
V	94.20		+0.88	+1.68	-2.52	+0.02
VI	95.52		-0.48	-0.52	+0.52	+0.16
VII	96.04		+0.08	+1.52	-1.60	0.00

TABLE 66.

VALUES OF THE U's FOR THE DETERMINATION OF THE JUST IMPERCEPTIBLE NEGATIVE
DIFFERENCE.

	I	II	III	IV	V	VI	VII
84	0.0644	0.0667	0.0044	0.0400	0.0400	0.0267	0.0200
88	0.1351	0.1286	0.0442	0.0736	0.0704	0.1331	0.1012
92	0.3362	0.2414	0.1247	0.2068	0.2669	0.2380	0.2285
96	0.3085	0.3104	0.2480	0.3806	0.4421	0.3071	0.3533
100	0.1368	0.1944	0.3588	0.2362	0.1487	0.2341	0.2178
104	0.0188	0.0528	0.2067	0.0596	0.0309	0.0569	0.0745
108	0.0003	0.0056	0.0128	0.0031	0.0010	0.0040	0.0046
Σ	1.0001	0.9999	0.9996	0.9999	1.0000	0.9999	0.9999
R	0.0000	0.0001	0.0004	0.0001	0.0000	0.0001	0.0002

TABLE 67.
VALUES OF $r_k U_k$ FOR THE DETERMINATION OF THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE.

r_k	I	II	III	IV	V	VI	VII
84	5.4096	5.6028	0.3696	3.3600	3.3600	2.2428	1.6800
88	11.8888	11.3168	3.8896	6.4768	6.1952	11.7128	8.9056
92	30.9304	22.2088	11.4724	19.0256	24.5548	21.8060	21.0220
96	29.6160	29.7984	23.8080	36.5376	42.4416	29.4816	33.9168
100	13.6765	19.4420	35.8775	23.6240	14.8685	23.4094	21.7780
104	1.9552	5.4912	21.4968	6.1984	3.2136	5.9176	7.7480
108	0.0324	0.6048	1.3824	0.3348	0.1080	0.4320	0.4968
Σ	93.5089	94.4648	98.2963	95.5572	94.7417	95.0922	95.5472

TABLE 68.
VALUES OF $r_k U_k$ FOR THE DETERMINATION OF THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE.

r_k	I	II	III	IV	V	VI	VII
84	454.4064	470.6352	31.0464	282.2400	282.2400	188.3952	141.1200
88	1046.2144	995.8784	342.2848	569.9584	545.1776	1080.7264	783.6928
92	2845.5968	2043.2096	1055.4608	1750.3552	2259.0416	2014.4320	1934.0240
96	2843.1380	2860.6464	2285.5680	3507.6096	4074.3936	2830.2336	3256.0128
100	1367.6500	1944.2000	3587.7500	2382.4000	1486.8500	2340.9400	2177.8000
104	203.3408	571.0848	2235.6672	644.6336	334.2144	615.4304	805.7920
108	3.4992	65.3184	149.2992	36.1584	11.6640	46.6560	53.6544
Σ	8763.8436	8950.9728	9687.0764	9153.3552	8993.5812	9066.8136	9152.0960

TABLE 69.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT I.

	IVa		I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	7	588	6	504	7	588	5	420
88	12	1056	13	1144	9	792	8	704
92	33	3036	50	4600	33	3036	18	1656
96	39	3744	18	1728	36	3456	40	3840
100	9	900	12	1200	13	1300	23	2300
104			1	104	2	208	6	624
Σ	100	9324	100	9280	100	9380	100	9544
Average		93.24		92.80		93.80		95.44

TABLE 70.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT II.

	IVa		I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	5	420	8	672	3	252	9	756
88	12	1056	14	1232	13	1144	12	1056
92	19	1748	23	2116	21	1932	28	2576
96	35	3360	26	2496	38	3648	31	2976
100	19	1900	22	2200	15	1500	15	1500
104	9	936	5	520	10	1040	4	416
108	1	108	2	216			1	108
Σ	100	9528	100	9452	100	9516	100	9388
Average		95.28		94.52		95.16		93.88

TABLE 71.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT III.

	IVa		I		III		IV	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	2	168						-
88	8	704	5	440				
92	14	1288	6	552	3	276	11	1012
96	24	2304	21	2016	31	2976	41	3936
100	33	3300	47	4700	61	6100	36	3600
104	18	1872	20	2080	2	208	10	1040
108	1	108	1	108	3	324	2	216
Σ	100	9744	100	9896	100	9884	100	9804
Average		97.44		98.96		98.84		98.04

TABLE 72.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT IV.

	I		III		fW	
r_k	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	1	84	4	336	5	420
88	9	792	7	616	1	88
92	15	1380	31	2852	11	1012
96	46	4416	36	3456	39	3744
100	21	2100	17	1700	28	2800
104	8	832	5	520	14	1456
108					2	216
Σ	100	9604	100	9480	100	9736
Average		96.04		94.80		97.36

TABLE 73.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT V.

r_k	I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	2	168	2	168	7	588
88	12	1056	3	264	8	704
92	22	2024	21	1932	31	2852
96	38	3648	44	4224	50	4800
100	22	2200	23	2300	4	400
104	4	416	7	728		
108						
Σ	100	9512	100	9616	100	9344
Average		95.12		96.16		93.44

TABLE 74.

OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. SUBJECT VI.

r_k	I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	7	588				
88	16	1408	13	1144	9	792
92	36	3312	16	1472	25	2300
96	19	1824	39	3744	35	3360
100	18	1800	29	2900	23	2300
104	4	416	3	312	6	624
108					2	216
Σ	100	93.48	100	9572	100	9592
Average		93.48		95.72		95.92

TABLE 76.
OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. COMBINED RESULTS.

r_k	I		II		III		IV		V		VI		VII	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84	25	2100	25	2100	2	168	10	840	11	924	7	588	8	704
88	42	3696	51	4488	13	1144	17	1496	23	2024	38	3344	47	4324
92	134	12328	91	8372	34	3128	57	5244	74	6808	77	7084	113	10848
96	133	12768	130	12480	117	11232	121	11616	132	12672	93	8928	101	10100
100	57	5700	71	7100	177	17700	66	6600	49	4900	70	7000	23	2392
104	9	936	28	2912	50	5200	27	2808	11	1144	13	1352	8	864
108			4	432	7	756	2	216			2	216		
Σ	400	37528	400	37884	400	39328	300	28820	300	28472	300	28512	300	29232
Average		93.82		94.71		98.32		96.07		94.91		95.04		97.44

TABLE 75 WILL BE FOUND ON THE FOLLOWING PAGE

TABLE 75.
OBSERVATIONS ON THE JUST IMPERCEPTIBLE NEGATIVE DIFFERENCE. TABLE VII.

r_k	I		III		IV	
	N_k	$r_k N_k$	N_k	$r_k N_k$	N_k	$r_k N_k$
84						
88	2	176			6	528
92	12	1104	16	1472	19	1748
96	37	3552	32	3072	44	4224
100	32	3200	42	4200	27	2700
104	13	1352	7	728	3	312
108	4	432	3	324	1	108
Σ	100	9816	100	9796	100	9620
Average		98.16		97.96		96.20

TABLE 76 WILL BE FOUND ON THE PRECEDING PAGE

TABLE 77.
DIFFERENCES BETWEEN OBSERVED AND CALCULATED VALUES OF THE JUST IMPERCEPTIBLE
NEGATIVE DIFFERENCE.

Subject	Difference between observed and calculated values.					
	Calculated Value	IVa	I	III	IV	Totals
I	93.51	-0.27	-0.71	+0.29	+1.93	+0.31
II	94.46	+0.82	+0.06	+0.70	-0.58	+0.25
III	98.30	-0.86	+0.66	+0.54	-0.26	+0.02
IV	95.56		+0.48	-0.76	+1.80	+0.51
V	94.74		+0.38	+1.42	-1.30	+0.17
IV	95.09		-1.61	+0.63	+0.83	+0.05
VII	95.55		+2.61	+2.41	+0.65	+1.89

TABLE 78.
DEVIATIONS FROM THE THEORETICAL RESULTS FOR THE JUST PERCEPTIBLE AND THE JUST IMPERCEPTIBLE DIFFERENCES.

	Just Perceptible Positive Difference			Just Imperceptible Positive Difference			Just Perceptible Negative Difference			Just Imperceptible Negative Difference		
Subject I	Theoretical value 99.764			Theoretical value 99.139			Theoretical value 93.085			Theoretical value 93.509		
	Series	Observed value	Difference	Observed value	Difference	Observed value	Difference	Observed value	Difference	Observed value	Difference	
	IVa	101.120	+1.356	98.920	-0.219	91.800	-1.285	93.240	-0.269			
	I	100.320	+0.556	99.200	+0.061	92.280	-0.805	92.800	-0.709			
	III	100.000	+0.236	98.880	-0.259	94.430	+1.345	93.800	+0.291			
	IV	100.000	+0.236	98.360	-0.779	94.140	+1.055	95.440	+1.939			
Average		0.896	Average		0.337	Average		1.125	Average		0.802z	
Subject II	Theoretical value 98.305			Theoretical value 99.364			Theoretical value 95.278			Theoretical value 94.465		
	IVa	99.314	+1.009	99.560	+0.196	95.960	+0.682	95.280	+0.815			
	I	98.021	-0.289	99.320	-0.044	96.160	+0.182	94.520	-0.055			
	III	99.120	+0.815	98.080	-1.284	95.280	+0.002	95.160	+0.695			
	IV	98.586	+0.281	97.880	-1.484	93.560	-1.718	93.880	-0.585			
	Average		0.578	Average		0.752	Average		0.646	Average		0.537
Subject III	Theoretical value 99.816			Theoretical value 98.752			Theoretical value 97.404			Theoretical value 98.296		
	IVa	99.080	-0.736	98.040	-0.712	96.520	-0.884	97.440	-0.856			
	I	96.880	-2.936	99.800	+1.048	97.640	+0.236	98.960	+0.664			
	III	101.600	+1.784	100.440	+1.688	97.840	+0.436	98.840	+0.544			
	IV	100.440	+0.624	100.400	+1.648	97.800	+0.396	98.040	-0.256			
	Average		1.520	Average		1.274	Average		0.488	Average		0.580

	Just Perceptible Positive Difference		Just Imperceptible Positive Difference		Just Perceptible Negative Difference		Just Imperceptible Negative Difference	
	Theoretical value 97.979		Theoretical value 98.187		Theoretical value 95.228		Theoretical value 95.557	
Subject IV	I	98.680	+0.701	-0.067	94.320	-0.908	96.040	+0.483
	III	97.640	-0.359	-0.267	95.030	-0.198	94.800	-0.757
	IV	99.420	+1.261	-0.328	95.800	+0.572	97.360	+1.803
	Average 0.777		Average 0.221		Average 0.559		Average 1.014	
Subject V	Theoretical value 96.914		Theoretical value 97.370		Theoretical value 94.197		Theoretical value 94.742	
	I	96.240	+2.326	+2.950	95.080	+0.883	95.120	+0.378
	III	97.600	+0.686	-0.370	95.880	+1.683	96.160	+1.418
	IV	96.240	-0.674	-3.690	91.680	-2.517	93.440	-1.302
	Average 1.229		Average 2.337		Average 1.694		Average 1.033	
Subject VI	Theoretical value 99.208		Theoretical value 100.518		Theoretical value 95.522		Theoretical value 95.092	
	I	97.819	-1.349	-1.278	95.040	-0.482	93.480	-1.512
	III	101.240	+2.032	-0.838	95.000	-0.522	95.720	+0.628
	IV	102.210	+3.002	+1.322	96.040	+0.518	95.920	+0.828
	Average 2.141		Average 1.146		Average 0.507		Average 0.980	
Subject VII	Theoretical value 98.759		Theoretical value 100.959		Theoretical value 96.038		Theoretical value 95.547	
	I	101.680	+2.921	-2.279	96.120	+0.092	98.160	+2.613
	III	100.792	+2.033	-2.898	97.560	+1.522	97.960	+2.413
	IV	100.120	+1.361	-2.599	94.440	-1.598	96.200	-0.653
	Average 2.105		Average 2.592		Average 1.071		Average 1.893	

TABLE 79.

DEVIATIONS OF OBSERVED VALUES FROM THE THEORETICAL RESULTS FOR THE THRESHOLDS.

Threshold in direction of increase.				Threshold in direction of decrease.		
Subject I	Theoretical value 99.452			Theoretical value 93.297		
	Series	Observed value	Difference	Observed value	Difference	
	IVa	100.020	+0.568	92.520	-0.777	
	I	99.780	+0.308	92.540	-0.757	
	III	99.440	-0.012	94.115	+0.818	
	IV	99.180	-0.272	94.770	+1.473	
Average			0.290	Average 0.956		
Subject II	Theoretical value 98.835			Theoretical value 94.871		
	IVa	99.437	+0.602	95.620	+0.749	
	I	98.670	-0.165	95.340	+0.469	
	III	98.600	-0.235	95.220	+0.349	
	IV	98.233	-0.652	93.720	-2.151	
	Average			0.414	Average 0.929	
Subject III	Theoretical value 99.284			Theoretical value 97.850		
	IVa	98.560	-0.724	96.980	-0.870	
	I	98.340	-0.944	98.300	+0.550	
	III	101.020	+1.736	98.340	+0.490	
	IV	100.420	+1.144	97.920	+0.070	
	Average			1.137	Average 0.495	
Subject IV	Theoretical value 98.083			Theoretical value 95.393		
	I	98.400	+0.317	95.180	-0.213	
	III	97.780	-0.303	94.915	-0.478	
	IV	98.550	+0.467	96.580	+1.187	
	Average			0.362	Average 0.626	
	Subject V	Theoretical value 97.142			Theoretical value 94.469	
I		99.780	+2.638	95.100	+0.631	
III		97.300	+0.158	96.020	+1.551	
IV		94.960	-2.182	92.560	-1.909	
Average			1.659	Average 1.364		
Subject VI		Theoretical value 99.863			Theoretical value 95.307	
	I	98.530	-1.333	94.260	-1.147	
	III	100.460	+0.597	95.360	+0.053	
	IV	102.025	+2.162	95.980	+0.673	
	Average			1.364	Average 0.624	
	Subject VII	Theoretical value 99.859			Theoretical value 95.793	
I		100.180	+0.321	97.140	+1.347	
III		99.426	-0.433	97.760	+1.967	
IV		99.240	-0.619	95.320	-0.473	
Average			0.458	Average 1.262		

TABLE 80.
THEORETICAL VALUES OF THE THRESHOLDS.

Subject	Threshold in direction of increase	Threshold in direction of decrease	Interval of uncertainty
I	99.45	93.30	6.15
II	98.83	94.87	3.96
III	99.28	97.85	1.43
IV	98.08	95.39	2.69
V	97.14	94.47	2.67
IV	99.86	95.31	4.56
VII	99.86	95.79	4.07

TABLE 81.
OBSERVED VALUES OF THE THRESHOLDS.

Subject	Threshold in direction of increase	Threshold in direction of decrease	Interval of uncertainty
I	99.60	93.49	6.11
II	98.71	94.98	3.74
III	99.58	97.88	1.70
IV	98.24	95.56	2.68
V	97.34	94.57	2.78
IV	100.33	95.20	5.12
VII	99.62	96.74	2.88

TABLE 82.
RATIOS OF THE NUMBERS OF "HEAVIER-GUESS" AND "LIGHTER-GUESS" JUDGMENTS.

	I	II	IV	V	VI	VII
84	0.3333	0.6667	0.2500	0.3333	1.0000	0.0000
88	0.2727	0.8214	1.5000	2.7500	0.5455	0.4500
92	0.6021	1.5758	0.5909	1.2105	1.7500	0.6571
96	1.0101	1.2745	0.8485	4.9000	0.8302	0.9649
100	3.8605	1.3478	1.0769	5.1429	1.3830	1.4054
104	2.7273	1.5000	3.0000	28.0000	3.0714	2.6667
108	3.8000	1.2500	4.0000	10.0000	2.2222	4.2500

TABLE 83.
RELATIVE FREQUENCIES OF "G" JUDGMENTS.

Comparison Weight	I	II	III	IV	V	VI	VII
84	0.0622	0.0444	0.0044	0.0167	0.0133	0.0200	0.0133
88	0.1244	0.1133	0.0200	0.0500	0.0500	0.1133	0.0967
92	0.3311	0.1889	0.0600	0.1167	0.1400	0.2200	0.1933
96	0.4422	0.2578	0.0778	0.2033	0.1967	0.3233	0.3733
100	0.4644	0.2400	0.1489	0.1800	0.1433	0.3733	0.2967
104	0.0911	0.0889	0.0467	0.0667	0.0967	0.1900	0.1833
108	0.0533	0.0800	0.0156	0.0500	0.0367	0.0967	0.1400

TABLE 84.
ABSOLUTE FREQUENCY OF "G" JUDGMENTS.

f_k	I		II		III		IV		V		VI		VIII	
	N_k	$N_k f_k$	N_k	$N_k f_k$	N_k	$N_k f_k$	N_k	$N_k f_k$	N_k	$N_k f_k$	N_k	$N_k f_k$	N_k	$N_k f_k$
84	28	235.2	20	1680	2	168	5	420	1	336	6	504	4	336
88	60	5280	51	4518	9	792	15	1320	15	1320	34	2992	20	2552
92	140	12708	85	7820	27	2484	35	3220	42	3864	66	6072	58	5336
10	190	19108	110	11130	35	3300	61	5856	59	5664	97	9312	112	10752
600	209	20900	108	10800	67	6700	54	5400	43	4300	112	11200	80	8000
104	41	4284	40	4160	21	2108	25	2480	39	3016	57	5628	55	5720
108	24	2592	36	3888	7	756	10	1020	11	1158	20	3132	42	4536
Σ	708	67848	456	43972	169	16428	205	19016	203	19688	401	39140	380	38132
Average		96.10		96.43		97.78		97.15		96.98		97.65		98.03

TABLE 85.
DATA FOR THE DETERMINATION OF THE PSYCHOMETRIC FUNCTIONS.

r_k	I			II			III			IV		
	G	H	L	G	H	L	G	H	L	G	H	L
84	0.0622	0.0022	0.9356	0.0444	0.0222	0.9333	0.0044	0.0000	0.9956	0.0167	0.0233	0.9600
88	0.1244	0.0200	0.8556	0.1133	0.0244	0.8622	0.0200	0.0244	0.9556	0.0500	0.0267	0.9233
92	0.3311	0.0889	0.5800	0.1889	0.1111	0.7000	0.0600	0.0711	0.8689	0.1167	0.1167	0.7667
96	0.4422	0.2222	0.3356	0.2578	0.2933	0.4489	0.0778	0.2222	0.7000	0.2033	0.3567	0.4400
100	0.4644	0.4133	0.1222	0.2400	0.5289	0.2311	0.1489	0.4711	0.3800	0.1800	0.6100	0.2100
104	0.0911	0.8956	0.0133	0.0689	0.8166	0.0956	0.0467	0.8933	0.0600	0.0667	0.8833	0.0500
108	0.0533	0.9400	0.0067	0.0600	0.9044	0.0156	0.0156	0.9578	0.0267	0.0500	0.9300	0.0200

r_k	V			VI			VII		
	G	H	L	G	H	L	G	H	L
84	0.0133	0.0267	0.9600	0.0200	0.0067	0.9733	0.0133	0.0067	0.9800
88	0.0500	0.0233	0.9267	0.1133	0.0233	0.8633	0.0967	0.0067	0.8967
92	0.1400	0.1600	0.7000	0.2200	0.0633	0.7167	0.1933	0.0667	0.7400
96	0.1967	0.5133	0.2900	0.3233	0.1867	0.4900	0.3733	0.1700	0.4567
100	0.1433	0.6800	0.1767	0.3733	0.4200	0.2067	0.2967	0.4367	0.2667
104	0.0967	0.8700	0.0333	0.1900	0.7433	0.0667	0.1833	0.7567	0.0600
108	0.0367	0.9433	0.0200	0.0967	0.8633	0.0200	0.1400	0.8200	0.0400

TABLE 86.
VALUES OF THE PSYCHOMETRIC FUNCTIONS. SUBJECT I.

	G	H	L
84	0.0622	0.0022	0.9356
85	-0.0167	0.0433	0.9734
86	-0.0039	0.0434	0.9605
87	0.0532	0.0302	0.9166
88	0.1244	0.0200	0.8556
89	0.1926	0.0205	0.7869
90	0.2501	0.0337	0.7162
91	0.2955	0.0577	0.6468
92	0.3311	0.0889	0.5800
93	0.3608	0.1232	0.5160
94	0.3881	0.1576	0.4543
95	0.4152	0.1905	0.3943
96	0.4422	0.2222	0.3356
97	0.4664	0.2554	0.2782
98	0.4827	0.3020	0.2153
99	0.4846	0.3453	0.1701
100	0.4644	0.4133	0.1223
101	0.4157	0.5033	0.0810
102	0.3347	0.6168	0.0485
103	0.2232	0.7513	0.0255
104	0.0911	0.8956	0.0133
105	-0.0398	1.0294	0.0104
106	-0.1328	1.1186	0.0142
107	-0.1284	1.1125	0.0159
108	0.0533	0.9400	0.0067

TABLE 87.
VALUES OF THE PSYCHOMETRIC FUNCTIONS. SUBJECT II.

r_k	G	H	L
84	0.0444	0.0222	0.9334
85	0.0614	0.0285	0.9101
86	0.0786	0.0258	0.8956
87	0.0959	0.0227	0.8814
88	0.1133	0.0244	0.8623
89	0.1310	0.0337	0.8353
90	0.1497	0.0516	0.7987
91	0.1690	0.0777	0.7533
92	0.1889	0.1111	0.7000
93	0.2088	0.1505	0.6407
94	0.2279	0.1946	0.5775
95	0.2447	0.2424	0.5129
96	0.2578	0.2933	0.4489
97	0.2654	0.3471	0.3875
98	0.2659	0.4040	0.3301
99	0.2578	0.4643	0.2779
100	0.2400	0.5289	0.2311
101	0.2125	0.5975	0.1900
102	0.1761	0.6698	0.1541
103	0.1334	0.7436	0.1230
104	0.0889	0.8156	0.0955
105	0.0495	0.8795	0.0710
106	0.0251	0.9255	0.0494
107	0.0294	0.9403	0.0303
108	0.0800	0.9044	0.0156

TABLE 88.
VALUES OF THE PSYCHOMETRIC FUNCTIONS. SUBJECT III.

r_k	G	H	L
84	0.0044	0.0000	0.9956
85	-0.0398	0.0359	1.0038
86	-0.0347	0.0403	0.9944
87	-0.0089	0.0325	0.9764
88	0.0200	0.0244	0.9556
89	0.0435	0.0226	0.9339
90	0.0559	0.0299	0.9142
91	0.0612	0.0464	0.8924
92	0.0600	0.0711	0.8689
93	0.0577	0.1022	0.8401
94	0.0590	0.1381	0.8029
95	0.0647	0.1781	0.7573
96	0.0778	0.2222	0.7000
97	0.0960	0.2714	0.6326
98	0.1170	0.3278	0.5552
99	0.1364	0.3936	0.4700
100	0.1489	0.4711	0.3800
101	0.1490	0.5619	0.2891
102	0.1325	0.6655	0.2020
103	0.0975	0.7786	0.1239
104	0.0667	0.8933	0.0400
105	-0.0108	0.9955	0.0153
106	-0.0571	1.0636	-0.0065
107	-0.0629	1.0657	-0.0028
108	0.0156	0.9578	0.0266

TABLE 89.
VALUES OF THE PSYCHOMETRIC FUNCTIONS. SUBJECT IV.

r_k	G	H	L
84	0.0167	0.0233	0.9600
85	0.0386	0.0532	0.9082
86	0.0453	0.0502	0.9045
87	0.0469	0.0370	0.9161
88	0.0500	0.0267	0.9233
89	0.0581	0.0276	0.9143
90	0.0725	0.0430	0.8845
91	0.0926	0.0736	0.8338
92	0.1167	0.1167	0.7666
93	0.1425	0.1697	0.6878
94	0.1672	0.2294	0.6034
95	0.1883	0.2925	0.5192
96	0.2033	0.3567	0.4400
97	0.2106	0.4205	0.3689
98	0.2092	0.4839	0.3069
99	0.1986	0.5464	0.2550
100	0.1800	0.6100	0.2100
101	0.1548	0.6756	0.1696
102	0.1253	0.7439	0.1308
103	0.0948	0.8142	0.0910
104	0.0667	0.8833	0.0500
105	0.0448	0.9446	0.0106
106	0.0328	0.9863	-0.0191
107	0.0337	0.9904	-0.0241
108	0.0500	0.9300	0.0200

TABLE 90.
VALUES OF THE PSYCHOMETRIC FUNCTIONS. SUBJECT V.

	G	H	L
84	0.0133	0.0267	0.9600
85	0.0290	0.1116	0.8594
86	0.0348	0.1031	0.8621
87	0.0401	0.0610	0.8989
88	0.0500	0.0233	0.9267
89	0.0663	0.0108	0.9229
90	0.0883	0.0313	0.8804
91	0.1138	0.0833	0.8029
92	0.1400	0.1600	0.7000
93	0.1637	0.2514	0.5849
94	0.1822	0.3465	0.4713
95	0.1935	0.4360	0.3705
96	0.1967	0.5133	0.2900
97	0.1917	0.5744	0.2339
98	0.1797	0.6200	0.2003
199	0.1626	0.6530	0.1844
100	0.1433	0.6800	0.1767
101	0.1248	0.7088	0.1664
102	0.1198	0.7473	0.1329
103	0.1003	0.8009	0.0988
104	0.0967	0.8700	0.0333
105	0.0967	0.9469	-0.0436
106	0.0946	1.0121	-0.1067
107	0.0800	1.0299	-0.1099
108	0.0367	0.9433	0.0200

TABLE 91.
VALUES OF THE PSYCHOMETRIC FUNCTIONS. SUBJECT VI.

r_k	G	H	L
84	0.0200	0.0067	0.9733
85	0.0235	0.0171	0.9594
86	0.0476	0.0207	0.9317
87	0.0800	0.0218	0.8982
88	0.1133	0.0233	0.8634
89	0.1440	0.0272	0.8288
90	0.1814	0.0347	0.7839
91	0.1962	0.0465	0.7573
92	0.2200	0.0633	0.7167
93	0.2442	0.0853	0.6705
94	0.2697	0.1129	0.6174
95	0.2965	0.1458	0.5577
96	0.3233	0.1867	0.4900
97	0.3479	0.2338	0.4183
98	0.3670	0.2883	0.3447
99	0.3768	0.3504	0.2728
100	0.3733	0.4200	0.2067
101	0.3532	0.4963	0.1505
102	0.3148	0.5779	0.1073
103	0.2591	0.6616	0.0793
104	0.1900	0.7433	0.0667
105	0.1181	0.8166	0.0653
106	0.0599	0.8728	0.0673
107	0.0408	0.9001	0.0591
108	0.0967	0.8833	0.0200

TABLE 92.
VALUES OF THE PSYCHOMETRIC FUNCTIONS. SUBJECT VII.

r_k	G	H	L
84	0.0133	0.0067	0.9800
85	0.1190	-0.0242	0.9052
86	0.1366	-0.0252	0.8886
87	0.1181	-0.0119	0.8938
88	0.0967	0.0067	0.8966
89	0.0903	0.0248	0.8849
90	0.1059	0.0406	0.8535
91	0.1422	0.0538	0.8040
92	0.1933	0.0667	0.7400
93	0.2504	0.0819	0.6677
94	0.3045	0.1024	0.5931
95	0.3474	0.1310	0.5216
96	0.3733	0.1700	0.4567
97	0.3793	0.2205	0.4002
98	0.3657	0.2828	0.3515
99	0.3362	0.3557	0.3081
100	0.2967	0.4376	0.2666
101	0.2550	0.5223	0.2227
102	0.2187	0.6078	0.1735
103	0.1941	0.6878	0.1181
104	0.1833	0.7567	0.0600
105	0.1816	0.8086	0.0098
106	0.1743	0.8386	-0.0129
107	0.1337	0.8430	0.0233
108	0.1400	0.8200	0.0400

TABLE 93.
CALCULATED AND OBSERVED VALUES OF THE THRESHOLD IN THE DIRECTION OF INCREASE

Subject	Result of Method of Just Perceptible Difference		Value found by Interpolation
	Observed	Calculated	
I	99.60	99.45	100.95
II	98.71	98.83	99.55
III	99.58	99.28	100.32
IV	98.24	98.08	98.26
V	97.35	97.14	95.83
VI	100.33	99.86	101.04
VII	99.63	99.86	100.74

TABLE 94.
CALCULATED AND OBSERVED VALUES OF THE THRESHOLD IN THE DIRECTION OF DECREASE

Subject	Result of Method of Just Perceptible Difference		Value found by Interpolation
	Observed	Calculated	
I	93.49	93.30	93.26
II	94.98	94.87	95.20
III	97.88	97.85	98.65
IV	95.56	95.39	95.24
V	94.57	94.47	93.75
VI	95.20	95.31	95.82
VII	96.74	95.79	95.33

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III
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BY
F. M. URBAN, PH.D.
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1. Spelling in the Elementary School: An Experimental and Statistical Investigation. 1902. 98 pp. 16 figures in the text and an appendix of 26 tables. Price: cloth, \$1.25; paper, \$1.00.

By OLIVER P. CORNMAN, PH.D., District Superintendent of Schools, Philadelphia.

"Dr. Cornman has made an extensive investigation of the ability of school children in Philadelphia schools in the subject of spelling, and studied his results with such care that his work deserves special attention." Professor W. H. Burnham. *The Pedagogical Seminary*. Vol. VIII. No. 4, Dec., 1906. p. 491.

2. The Sensation of Pain and the Theory of the Specific Sense Energies. 1902. viii, 87 pp., with one table and two figures in the text. Price: paper, \$1.00.

By ANNA J. MCKEAG, PH.D., Professor of Pedagogy, Wellesley College.

As the result of an experimental investigation of the reaction of a number of subjects to painful stimuli, the author first presents an analysis of the pain judgment, which is presented either as a general or unforced judgment, or as a judgment in which the subject is required to give expression to the perceived qualitative distinctness and discreteness of the sensation. The relation of pain and touch to various stimuli is given consideration. In an important chapter a resume of the history of the theory of the specific sense energies is given, with special reference to the sensation of pain, and an examination is made of the physiological basis of pain in the periphery and nerve centres.

3. The Application of Statistical Methods to the Problem of Psychophysics. 1908. 221 pp., including 94 tables. Price: paper, \$2.50.

By F. M. URBAN, PH.D., University of Pennsylvania, Philadelphia.

4. Two Experimental Studies of the Insane. I. The Train of Thought. II. Some Physiological Conditions Accompanying States of Depression (In press.)

By CLARA HARRISON TOWN, Resident Psychologist at Friends' Asylum for the Insane, Frankford, Philadelphia, Pa.

5. Statistics of Retardation. (In preparation.)

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